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The Forced Flow of a Rotating Viscous Liquid which is Heated from below

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THE FORCED FLOW OF A ROTATING VISCOUS LIQUID WHICH IS HEATED FROM BELOW

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A liquid is contained in a cylindrical vessel and is subject to heating on the horizontal base of the vessel. The problem of the forced flow arising from the heating has been investigated in the case when the heating function is symmetrically arranged about the central axis. It is found that the relative forced flow tends to become zonal in character when the vessel rotates at a sufficiently high angular velocity. This relative zonal motion is principally in the direction of the rotation except near the outer portion of the fluid where it is in the opposite direction, the former being 'westerlies', the latter 'easterlies'. The easterlies are due to the non-linear inertia terms in the equations of motion. This description of the velocity field is used because the experiment described above has considerable meteorological significance.

List of symbols introduced in the text

		Page or equation where first used
U	a representative fluid velocity in the atmosphere	p. 84
C_E	the equatorial surface speed	p. 84
U/C_E	Rossby number	p. 84
r, ϕ, z	cylindrical co-ordinates	p. 84
u_1, v_1, w_1	velocity components in cylindrical co-ordinates referred to fixed space axes	p. 84
u, v, w	velocity components in cylindrical co-ordinates referred to axes moving with angular velocity Ω	(1·7)
p_1	total pressure	(1·7)
p_0	hydrostatic pressure	(1·7)
p	departure from hydrostatic pressure	(1·7)
ρ_1	total density	(1·7)
$\rho_0; \rho$	constant density of liquid; departure from constant density	(1·7)
$T_1; T_0$	total temperature; constant temperature	(1·5)
F_r, F_ϕ, F_z	viscous terms in equations of motion	(1·1), (1·2), (1·3)
χ_1	divergence of the velocity vector	(1·4)

		Page or equation where first used
c_v	specific heat	(1·6)
k	thermometric conductivity	(1·6)
Φ_1	viscous dissipation	(1·6)
α	inverse of coefficient of cubical expansion	(1·5)
μ	viscosity	(1·1)
g	gravitational acceleration	(1·3)
Ω	angular velocity of dishpan	(1·7)
$U_1(z), V_1(z), W_1(z)$	vertical variations of (u, v, w)	(2·7)
$\rho_1(z), P_1(z), T_1(z)$	vertical variations of ρ, p and T	(2·7)
$\beta; \beta_s$	a parameter; the infinite set of β values	(2·8), (2·23)
$h; r_0$	depth of fluid (2·15); radius of cylinder	(2·22)
a, a_1, a_2	non-dimensional geometric parameters	(2·17) $a = \beta h$, $a_s = \beta_s h$
ξ	a non-dimensional height	p. 87, $z = h\xi$
H	a parameter	(2·15)
$\kappa, \kappa_1, \kappa_2, \lambda$	parameters of dimension velocity	(2·20)
A_s, B_s	constants of integration	(2·21)
$u_s(\xi), v_s(\xi), w_s(\xi)$	terms in the expansion of (u, v, w) in ascending powers of a	(2·24), (2·24), (3·33)
$Q(r)$	a heat distribution function	(2·36)
Q^*	net flow of heat through base of cylinder	(2·37)
R	rotation Reynolds number	(3·13)
ω	square root of R	(3·22)
C_1, S_1, c_1, s_1	$\cosh\omega$, $\sinh\omega$, $\cos\omega$, $\sin\omega$	(following (3·23))
δ	boundary-layer thickness	(3·27)
β_0	angle between velocity vector and isobars	(3·39)
v^*	zonal velocity v for large R	(3·42)
ψ	dimensionless Stokes stream function	(4·9)
η	dimensionless radial variable	(4·11)
ϵ	dimensionless parameter	(4·11)
ϵ^*	dimensionless parameter related to Rossby number	(4·11)
$F_0, F_1, A, B, C, D, \alpha_1, \beta_1, \gamma_1, \delta_1$	functions of η only	(4·16), (4·24), (6·9)
G_0, G_1	functions of ξ only	(4·18), (4·34)
ψ_s	terms in expansion of ψ in ascending powers of ϵ^*	(4·12)
Ψ_1	function of ξ only	(4·24)
$A', B', C', D', \alpha_1', \beta_1', \gamma_1'$	constants of integration	p. 101, (5·18)
$\alpha(\xi); \sigma$	see (4·18); a surface on which $v = 0$	p. 105
$\chi(r)$	an arbitrary function of r	(4·28)

1. Introduction

This work has been largely inspired by a certain experiment (Fultz 1951) which has been conducted recently at the University of Chicago and is an attempt to solve some of the hydrodynamical problems which it presents. In the experiment water is contained in a cylindrical vessel (a dishpan) of 15 cm radius to a depth of 2 cm, and the effects of heating the horizontal base of the vessel near the side have been observed for various rates of rotation of the vessel about its central axis. The heating is approximately symmetrical about the central point of the base, and the difference in temperature between the outer portions of the fluid and the central portions is usually between 5 and 15° C. Two principal régimes have emerged which are referred to as the low- and high-rotation cases. In the former the motion at the free surface is predominantly symmetrical about the central axis, and any variations in the transverse direction are small. In the latter, the motion is markedly asymmetrical in character, consisting of distinct finite-amplitude wave patterns in the transverse direction. With a fixed heating system at the base the transition from the low- to the high-rotation régimes occurs when the angular velocity of the dishpan attains a certain critical value and the change in the nature of the flow takes place rapidly at this value. The experiment acquires considerable interest, due to the fact that there are marked similarities between the high-rotation régime and observed flows of the earth's atmosphere.

In the present paper only the symmetrical forced flow will be discussed. It is convenient first of all to deal with the problem of zero rotation in order to establish a method of attack which exploits the fact that the fluid is shallow. We consider next the forced motion when the dishpan rotates, in order to obtain the essential mechanism of the low-rotation régime. This is first done approximately by omitting all the non-linear terms. These terms are shown to be small when the temperature difference between the axis and the rim is sufficiently small. Subsequently the non-linear terms are incorporated by using the method of small parameters.

It will be assumed throughout that the flow of the liquid relative to the dishpan is small compared with the flow of solid rotation, and that the flow of heat within the fluid takes place entirely by molecular conduction. The former assumption is quite valid for the experiment, but the latter assumption is justifiable only in the low-rotation régime when steady conditions persist.

When the non-linear terms in the equations are omitted it is shown that with a given symmetrical heating distribution on the base there is a certain forced symmetrical flow established relative to the rotating vessel. This symmetrical flow exists for all values of the rotation (which enters into the problem through a rotation Reynolds number R) and there is no possibility of breakdown of this flow contained within this solution. It appears that with the appropriate values which pertain to the low-rotation experiments ($R \approx 42$) the symmetrical flow on the linear theory consists essentially of a 'westerly' zonal flow which increases linearly with the height above the base (except in a thin boundary layer where the zonal velocity is proportional to the third power of the height) and whose radial variation depends upon a Bessel function of order one. The orders of magnitude of the velocity components in this case are approximately correct, and the observed zonal velocity maximum near 7·5 cm from the central axis is a feature of the solution. This zonal flow which develops for large R may be called the 'thermal wind' of the problem, since it arises in a similar way to that in meteorological theory. But even though some of the important features of the observed low-rotation flow are already present in this solution there are also some incorrect features. One such feature and one which could be anticipated is the absence of any horizontal stress at the base, this being essentially connected with the ignoring of the non-linear terms. With no heating at the base the liquid of course has no relative motion to the boundaries, but the present problem is so arranged that the total heat supplied to the liquid is zero. It is assumed that no heat escapes at the free surface and the curved sides of the liquid and that as much heat is withdrawn at the base as is supplied. These are ideal assumptions which will not be exactly satisfied in the experiment but nevertheless serve as a first approximation. In the corresponding problem for the earth's atmosphere, which has been borne in mind throughout the present problem, such assumptions are likely to be more valid. Using an angular momentum argument it follows that if the angular velocity of the vessel about its central axis is constant the supply and withdrawal of equal quantities of heat to the liquid will imply that the moment of the boundary stresses about the axis of rotation must vanish. Thus the zonal motion must contain both easterlies and westerlies. Stress upon the base is connected with the non-linear terms in the equations, and when these are introduced into the problem we might therefore expect to find both easterlies and westerlies in the solution for the zonal flow. In §§ 4 to 6 it is shown that they are actually present in the solution and that their existence is due solely to the non-linear terms. The introduction of these terms is effected by the method of expansion in powers of a non-dimensional parameter ϵ^* which is identical with the Rossby number, U/C_E, apart from a factor of 2. This non-dimensional number has been introduced into meteorological and model problems by Fultz (1949). The velocity field within the fluid has not yet been determined experimentally. It should be added that the stress of the air upon the free surface has been ignored throughout, otherwise the angular momentum argument for westerlies and easterlies is untenable.

The solution obtained here is not complete, since the conditions at the side walls are not exactly satisfied, but this is not of great importance, as may be noted from a result in §§ 2 and 3 relating to the position of the zonal velocity maximum. The exact symmetrical régime investigated here is unlikely to be reproduced in the laboratory due to variable roughness at the base of the vessel and to small departures from symmetry in the heating of the base. These effects are probably responsible for introducing small departures from symmetry in the low-rotation régime.

In setting up the problem cylindrical co-ordinates (r, ϕ, z) are used and there are six dependent variables $(u_1, v_1, w_1, p_1, \rho_1, T_1)$, where u_1 represents the velocity component in the direction r increasing, v_1 in the direction ϕ increasing, w_1 in the direction z increasing. Connecting these six dependent variables are the following six equations:

$$\rho_1\left(\frac{\mathrm{d}u_1}{\mathrm{d}t}-\frac{v_1^2}{r}\right) = -\frac{\partial p_1}{\partial r}+\mu\left(\nabla^2 u_1-\frac{u_1}{r^2}-\frac{2}{r^2}\frac{\partial v_1}{\partial \phi}\right)+\tfrac{1}{3}\mu\frac{\partial \chi_1}{\partial r} = -\frac{\partial p_1}{\partial r}+\mu F_r, \tag{1·1}$$

$$\rho_1\left(\frac{\mathrm{d}v_1}{\mathrm{d}t}+\frac{u_1 v_1}{r}\right) = -\frac{\partial p_1}{r\,\partial \phi}+\mu\left(\nabla^2 v_1-\frac{v_1}{r^2}+\frac{2}{r^2}\frac{\partial u_1}{\partial \phi}\right)+\tfrac{1}{3}\mu\frac{\partial \chi_1}{r\,\partial \phi} = -\frac{\partial p_1}{r\,\partial \phi}+\mu F_\phi, \tag{1·2}$$

$$\rho_1\frac{\mathrm{d}w_1}{\mathrm{d}t} = -\frac{\partial p_1}{\partial z}-g\rho_1+\mu\nabla^2 w_1+\tfrac{1}{3}\mu\frac{\partial \chi_1}{\partial z} = -\frac{\partial p_1}{\partial z}-g\rho_1+\mu F_z, \tag{1·3}$$

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}t}+\rho_1\chi_1 = 0, \quad \chi_1 = \frac{\partial u_1}{\partial r}+\frac{u_1}{r}+\frac{\partial v_1}{r\,\partial \phi}+\frac{\partial w_1}{\partial z}, \tag{1·4}$$

$$\rho_1 = \rho_0-\alpha(T_1-T_0), \tag{1·5}$$

$$\rho_1 J c_v\frac{\mathrm{d}T_1}{\mathrm{d}t}-p_1\chi_1 = Jk\nabla^2 T_1+\Phi_1, \tag{1·6}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial z^2}$. Equation (1·5) is an equation of state which is valid for liquids within sufficiently small temperature ranges. If we take this range to be from 20 to 30° C we have the following numerical data:

$$\begin{aligned} 20^\circ\text{ C} \quad \rho_0 &= 0{\cdot}998,2343,\\ 30^\circ\text{ C} \quad \rho_1 &= 0{\cdot}995,6780,\\ \alpha &= 0{\cdot}000,25563. \end{aligned}$$

α is the inverse of the coefficient of cubical expansion. The viscosity μ and the specific heat c_v vary with temperature, but these effects cannot be incorporated. Equation (1·6) is the equation of heat transfer (see Goldstein, *Modern developments in fluid dynamics*, **2**, 603) which contains the convection, divergence, conduction and dissipation terms.

With no heating in the experiment, rotation of the dishpan produces solid rotation of the water; with heating the velocity field shows a departure from solid rotation, but these departures from solid rotation are always less in magnitude than the flow of solid rotation.

Suppose that suffix zero indicates the steady state of solid rotation and let Ω be the angular velocity of the dishpan about its central axis, then if we write

$$u_1 = u, \quad v_1 = r\Omega + v, \quad w_1 = w, \quad \rho_1 = \rho_0 + \rho, \quad p_1 = p_0 + p, \quad T_1 = T_0 + T, \tag{1·7}$$

the equations governing the basic steady flow of solid rotation will be

$$\left.\begin{aligned} -\frac{\partial p_0}{\partial r} &= -\rho_0 r\Omega^2,\\ \frac{\partial p_0}{r\,\partial \phi} &= 0,\\ \frac{\partial p_0}{\partial z} &= -g\rho_0. \end{aligned}\right\} \tag{1·8}$$

The rotation speeds in the experiment vary from 1 to 4 revolutions per minute from 'low' to 'high' rotation respectively, so that Ω varies approximately from 0·1 to 0·4 radian per second. This has the effect of making $r\Omega^2/g$ have the maximum value of about 0·007, and it follows that the curvature of the free upper surface of the fluid can therefore be ignored. Hereafter the free surface will be taken to be the horizontal surface $z = h$, and the bottom of the dishpan to be $z = 0$.

The equations governing the symmetrical departures from the steady state of solid rotation will be as follows:

$$\rho_0\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}-2\Omega v\right)-\rho r\Omega^2 = -\frac{\partial p}{\partial r}+\mu F_r, \tag{1·9}$$

$$\rho_0\left(u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}+2\Omega u\right) = \mu F_\phi, \tag{1·10}$$

$$\rho_0\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z}-g\rho+\mu F_z, \tag{1·11}$$

$$u\frac{\partial \rho}{\partial r}+w\frac{\partial \rho}{\partial z}+\rho_0\left(\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}\right) = 0, \tag{1·12}$$

$$\rho = -\alpha T, \tag{1·13}$$

$$\rho_0 Jc_v\left(u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right)-p_0\left(\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}\right) = Jk\nabla^2 T+\Phi_1. \tag{1·14}$$

In the first equation if we use approximate values from the experiment $v = 1$ cm/s, $\rho_0 = 1$, $\Omega = \frac{1}{2}$, we obtain $2\rho_0 v = 2$ and $\rho r\Omega = 0{\cdot}0187$, hence we shall ignore henceforth the term $\rho r\Omega^2$ in (1·9). It may be noted that the dissipation term Φ_1 which appears in (1·14) is of the second degree in the velocity components.

As stated earlier we now consider in succession three cases of the above equations:

(*a*) zero rotation, with heating, no variations in time or ϕ. No non-linear terms;

(*b*) rotation Ω, with heating, no variations in time or ϕ. No non-linear terms;

(*c*) influence of non-linear terms upon (*b*).

2. Zero rotation, no variations in time or ϕ. No non-linear terms

The equations for the departure motion (1·9) to (1·14) now become

$$0 = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u - \frac{u}{r^2}\right), \tag{2·1}$$

$$0 = \mu\left(\nabla^2 v - \frac{v}{r^2}\right), \tag{2·2}$$

$$0 = -\frac{\partial p}{\partial z} - g\rho + \mu\nabla^2 w, \tag{2·3}$$

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{2·4}$$

$$\rho = -\alpha T, \tag{2·5}$$

$$0 = \nabla^2 T, \tag{2·6}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$. The normal type of solution of these equations from which the general solution can be derived may be obtained by making the following substitutions in the above equations:

$$\left.\begin{aligned} u &= U_1(z)\, J_1(\beta r), \\ v &= 0, \\ w &= W_1(z)\, J_0(\beta r), \\ T &= T_1(z)\, J_0(\beta r), \\ \rho &= \rho_1(z)\, J_0(\beta r), \\ p &= P_1(z)\, J_0(\beta r). \end{aligned}\right\} \tag{2·7}$$

Here β is a constant which is at our disposal and $J_0(x)$, $J_1(x)$ are the Bessel functions of orders zero and one respectively. When we make use of the results

$$\left.\begin{aligned} \left(\nabla^2 - \frac{1}{r^2}\right) U_1(z)\, J_1(\beta r) &= J_1(\beta r)\,\{U_1'' - \beta^2 U_1\}, \\ \nabla^2 W_1(z)\, J_0(\beta r) &= J_0(\beta r)\,\{W_1'' - \beta^2 W_1\}, \\ \frac{d}{dr} J_0(\beta r) &= -\beta J_1(\beta r), \\ \frac{d}{dr}\{r J_1(\beta r)\} &= \beta r J_0(\beta r), \end{aligned}\right\} \tag{2·8}$$

it is easily shown that the above equations are reduced to the following system of ordinary differential equations in U_1, W_1, T_1, P_1 and ρ_1:

$$0 = \beta P_1 + \mu(U_1'' - \beta^2 U_1), \tag{2·9}$$

$$0 = -P_1' + g\alpha T_1 + \mu(W_1'' - \beta^2 W_1), \tag{2·10}$$

$$\beta U_1 + W_1' = 0, \tag{2·11}$$

$$T_1'' - \beta^2 T_1 = 0. \tag{2·12}$$

If we eliminate P_1 between (2·9) and (2·10) we obtain

$$T_1 = \frac{\mu}{\beta^2 g\alpha}\{W_1^{\text{iv}} - 2\beta^2 W_1'' + \beta^4 W_1\}, \tag{2·13}$$

and it follows then from (2·12) that W_1 satisfies the equation

$$(\mathrm{D}_1^2 - \beta^2)^3 W_1 = 0, \quad \mathrm{D}_1 \equiv \frac{\mathrm{d}}{\mathrm{d}z}. \tag{2·14}$$

Let us consider now the boundary conditions in the problem. At the bottom of the fluid the velocity components u and w must vanish, and we shall assume the heating is prescribed there also by taking the heating function on the bottom to be $\partial T/\partial z = HJ_0(\beta r)$, where H is a constant. At the free surface we shall assume that the fluid is subject to no stresses and this implies that $w = 0$ and $\partial u/\partial z = 0$ at $z = h$. Here also we shall assume that there is no heat flow across the free surface so that $\partial T/\partial z = 0$ at $z = h$. Hence we have the following six boundary conditions:

$$\left.\begin{aligned} &W_1(0) = 0, \quad U_1(0) = 0, \quad \left.\frac{\mathrm{d}T_1}{\mathrm{d}z}\right|_{z=0} = H, \\ &W_1(h) = 0, \quad U_1'(h) = 0, \quad \left.\frac{\mathrm{d}T_1}{\mathrm{d}z}\right|_{z=h} = 0. \end{aligned}\right\} \tag{2·15}$$

These boundary conditions can all be expressed in terms of W_1 and its derivatives by using the equations (2·9) to (2·12), and we then obtain

$$\left.\begin{aligned} &W_1(0) = 0, && W_1(h) = 0, \\ &W_1'(0) = 0, && W_1''(h) = 0, \\ &W_1^{\text{v}}(0) - 2\beta^2 W_1'''(0) + \beta^4 W_1'(0) = \frac{H\beta^2 g\alpha}{\mu}, && W_1^{\text{v}}(h) - 2\beta^2 W_1'''(h) + \beta^4 W_1'(h) = 0. \end{aligned}\right\} \tag{2·16}$$

The set of equations (2·14) and (2·16) will then define $W_1(z)$ uniquely.

It is convenient in this problem to introduce a non-dimensional parameter a defined by

$$a = \beta h, \tag{2·17}$$

and to write $z = h\xi$. In this case the problem becomes

$$\left.\begin{aligned} &(\mathrm{D}^2 - a^2)^3 W_1 = 0, && \mathrm{D} \equiv \frac{\mathrm{d}}{\mathrm{d}\xi}, \\ &W_1(0) = 0, && W_1(1) = 0, \\ &W_1'(0) = 0, && W_1''(1) = 0, \\ &W_1^{\text{v}}(0) - 2a^2 W_1'''(0) + a^4 W_1'(0) = \frac{H\beta^2 g\alpha h^5}{\mu}, && W_1^{\text{v}}(1) - 2a^2 W_1'''(1) + a^4 W_1'(1) = 0. \end{aligned}\right\} \tag{2·18}$$

We consider first of all the exact solution of the problem presented by (2·18), and for this purpose it is more convenient to return to the equations (2·9) to (2·12). Thus we immediately obtain

$$T_1 = -\frac{hH\cosh a(1-\xi)}{a\sinh a}, \tag{2·19}$$

and the $W_1(z)$, equation (2·13), becomes

$$(\mathrm{D}^2-a^2)^2 W_1 = -\frac{ag\alpha h^3 H}{\mu \sinh a}\cosh a(1-\xi)$$
$$= -\frac{a\kappa}{\sinh a}\cosh a(1-\xi) = \kappa_1 \cosh a(1-\xi), \tag{2·20}$$

where $\kappa = g\alpha h^3 H/\mu$ has the dimensions of velocity. The complete solution for W_1 may be written in the form

$$W_1 = (A_0 + A_1\xi)\sinh a(1-\xi) + (A_2 + A_3\xi)\sinh a\xi + \frac{\kappa_1}{8a^2}\xi^2\cosh a(1-\xi).$$

When we use the conditions (2·18) to solve for the constants A_s we obtain the following solution for W_1:

$$\begin{aligned} -8a^2\sinh^2 a\left(\frac{a}{\sinh a}-\cosh a\right)W_1(\xi) = &-\frac{a\kappa}{\sinh a}(a\cosh a-\sinh a)\,\xi\sinh a(1-\xi) \\ &+\kappa(a\cosh a-\sinh a)\sinh a\xi \\ &-\kappa\left(\frac{a^2}{\sinh a}-\sinh a\right)\xi\sinh a\xi \\ &+\kappa a\sinh a\left(\frac{a}{\sinh a}-\cosh a\right)\xi^2\cosh a(1-\xi). \end{aligned} \tag{2·21}$$

The above solution, derived by the exact method, possesses two disadvantages: the nature of the variation of $W_1(\xi)$ with ξ is relatively obscure, and the corresponding solutions in later developments of this theory cannot be obtained in this way. Therefore before proceeding to a discussion of the solution (2·21) it is of more importance to establish an approximate method which will not only clarify the nature of the $W_1(\xi)$ but will also be applicable to later problems.

One of the important features of the experiment is the fact that the fluid depth is small compared with the radius of the dishpan, and the method we introduce makes use of this feature. It will be noted from (2·7) that if we are to satisfy the side condition

$$u = v = 0, \quad r = r_0, \tag{2·22}$$

it is necessary that we choose β so that

$$J_1(\beta r_0) = 0. \tag{2·23}$$

This will define an infinite set of β's which we shall denote by β_s $(s = 1, 2, 3, \ldots)$. For the present we confine our attention to the first zero of J_1 which gives

$$\beta_1 r_0 = 3·83.$$

Hence the parameter a defined in (2·17) will in this case have the value $a = 0·6$. The possibility therefore arises of expanding the solution for $W_1(\xi)$ in ascending powers of a^2 (since it is only the square of a which appears in (2·18)). It is evident from (2·18) that the leading term in such an expansion of W must commence with a term which is independent of a^2; thus we assume that

$$W_1(\xi) = w_0(\xi) + a^2 w_1(\xi) + a^4 w_2(\xi) + \ldots, \tag{2·24}$$

where the functions $w_s(\xi)$ are dependent upon ξ alone. If we use (2·11) we obtain the corresponding form of the solution for $U_1(\xi)$, namely,

$$U_1(\xi) = \frac{1}{a} u_0(\xi) + a u_1(\xi) + a^3 u_2(\xi) + \ldots, \tag{2·24'}$$

where the functions $u_s(\xi)$ are dependent upon ξ alone. When (2·24) is inserted in (2·18) and the successive coefficients of a^2 equated to zero, the following systems of equations are obtained. The first set contains $w_0(\xi)$ only

$$\left.\begin{array}{c} w_0^{\mathrm{vi}}(\xi) = 0, \\ w_0(0) = 0, \quad w_0(1) = 0, \\ w_0'(0) = 0, \quad w_0''(1) = 0, \\ w_0^{\mathrm{v}}(0) = 0, \quad w_0^{\mathrm{v}}(1) = 0; \end{array}\right\} \tag{2·25}$$

the next involves $w_1(\xi)$ and $w_0(\xi)$,

$$\left.\begin{array}{c} w_1^{\mathrm{vi}}(\xi) = 3w_0^{\mathrm{iv}}(\xi), \\ w_1(0) = 0, \qquad w_1(1) = 0, \\ w_1'(0) = 0, \qquad w_1''(1) = 0, \\ w_1^{\mathrm{v}}(0) = 2w_1'''(0) + \kappa, \quad w_1^{\mathrm{v}}(1) = 2w_1'''(1), \end{array}\right\} \tag{2·26}$$

and so on to sets which involve w_2, w_1 and w_0, etc. Thus we may solve successively for $w_0(\xi)$, $w_1(\xi)$, $w_2(\xi)$, ... The success of this method depends essentially upon the rapidity of convergence of the series (2·24). The solution for $w_0(\xi)$ is readily obtained and is merely the quartic

$$w_0(\xi) = \frac{A_4}{2\,.\,4!} \xi^2 (1-\xi)(3-2\xi), \tag{2·27}$$

where A_4 is a constant which is not determined at this stage due to two of the boundary conditions in (2·25) being identical. Proceeding to the solution for $w_1(\xi)$ from (2·26) the following situation arises. We must solve

$$w_1^{\mathrm{vi}}(\xi) = 3A_4,$$

subject to the stated boundary conditions. The general solution for $w_1(\xi)$ will be

$$w_1(\xi) = \frac{B_2}{2!}\xi^2 + \frac{B_3}{3!}\xi^3 + \frac{B_4}{4!}\xi^4 + \frac{B_5}{5!}\xi^5 + \frac{3A_4}{6!}\xi^6. \tag{2·28}$$

In order to satisfy $w_1^{\mathrm{v}}(0) = 2w_0'''(0) + \kappa$, we must have

$$B_5 = -\tfrac{5}{4}A_4 + \kappa,$$

and in order to satisfy $w_1^{\mathrm{v}}(1) = 2w_1'''(1)$ we must have

$$B_5 = -\tfrac{9}{4}A_4,$$

hence these two equations determine A_4 and B_5,

$$A_4 = -\kappa, \quad B_5 = \tfrac{9}{4}\kappa, \tag{2·29}$$

and thus the constant A_4 which is not determined by (2·25) becomes determinate at the second stage in order to make the set of equations (2·26) self-consistent. We thus obtain

$$w_0(\xi) = -\frac{\kappa}{2\,.\,4!}\xi^2(1-\xi)(3-2\xi), \tag{2·30}$$

and evaluating $w_1(\xi)$ we obtain

$$w_1(\xi) = \frac{B_4}{2\,.\,4!}\xi^2(1-\xi)(3-2\xi)+\frac{\kappa\xi^2}{960}(1-\xi)(39-14\xi-14\xi^2+4\xi^3). \tag{2·31}$$

Here B_4 is a constant which is not determined by (2·26) but is determinate at the third stage. This leads to $B_4 = -\frac{7}{12}\kappa$ and the solution (2·31) for $w_1(\xi)$ becomes considerably simplified. There is no necessity to proceed to any higher approximations, since when we combine the above results and substitute in (2·24) we obtain

$$W_1(\xi) = -\frac{\kappa\xi^2(1-\xi)(3-2\xi)}{48}\{1-\tfrac{1}{30}a^2(2+6\xi-3\xi^2)+O(a^4)\}. \tag{2·32}$$

The question now arises to what extent does this solution give a good approximation to the exact solution in (2·21). When the exact solution (2·21) is expanded in ascending powers of a^2 it has been verified that (2·32) is true to the stated order. Thus the approximate method is sound. It will be noted also that when we insert the value of $a^2 = 0{\cdot}36$ we find that the correction term in (2·32) is at most a 4 % correction of the leading term, hence the method is adequate for the discussion of the present type of problem.

In order to discuss the details of the solution it is sufficient to take the leading term in (2·32), and when we do this we obtain the approximate formulae for the velocity components u and w. These are given by

$$\left.\begin{aligned} u &= \frac{\kappa}{48a}\xi(6-15\xi+8\xi^2)\,J_1(\beta r)+O(a),\\ w &= -\frac{\kappa}{48}\xi^2(1-\xi)(3-2\xi)\,J_0(\beta r)+O(a^2).\end{aligned}\right\} \tag{2·33}$$

The side conditions $u = 0$, $v = 0$, $w = 0$ at $r = r_0$ have been partially considered already (2·22). It remains to satisfy $w = 0$ at $r = r_0$, and this we shall do approximately by considering only the leading terms of w in (2·33). If β_1 and β_2 are two solutions of (2·23) and κ_1, κ_2 are two heating constants, it follows that we may construct solutions for u and w of the form

$$u = \frac{\xi(6-15\xi+8\xi^2)}{48}\left\{\frac{\kappa_1}{a_1}J_1(\beta_1 r)+\frac{\kappa_2}{a_2}J_1(\beta_2 r)\right\},$$

$$w = -\frac{\xi^2(1-\xi)(3-2\xi)}{48}\{\kappa_1 J_0(\beta_1 r)+\kappa_2 J_0(\beta_2 r)\}.$$

If we now choose $\kappa_1 : \kappa_2$ so that

$$\frac{\kappa_1}{J_0(\beta_2 r_0)} = \frac{\kappa_2}{-J_0(\beta_1 r_0)} = \lambda,$$

we may satisfy the condition $w = 0$ at $r = r_0$. Hence we obtain the following solutions:

$$\frac{u}{\lambda} = \tfrac{1}{48}\xi(6-15\xi+8\xi^2)\left\{\frac{1}{a_1}J_0(\beta_2 r_0)\,J_1(\beta_1 r)-\frac{1}{a_2}J_0(\beta_1 r_0)\,J_1(\beta_2 r)\right\}, \tag{2·34}$$

$$\frac{w}{\lambda} = -\tfrac{1}{48}\xi^2(1-\xi)(3-2\xi)\{J_0(\beta_2 r_0)\,J_0(\beta_1 r)-J_0(\beta_1 r_0)\,J_0(\beta_2 r)\}. \tag{2·35}$$

The form of heating on the bottom which produces the above motion is evidently

$$Q(r) = \left.\frac{\partial T}{\partial z}\right|_{z=0} = \frac{\mu\lambda}{g\alpha h^3}\{J_0(\beta_2 r_0)\, J_0(\beta_1 r) - J_0(\beta_1 r_0)\, J_0(\beta_2 r)\}. \tag{2·36}$$

Thus there is an exact correlation between the vertical velocity w and the heating function on the base. It will be observed that with the present solution no heat is passing through the side $r = r_0$ of the vessel, and the net flow of heat Q^* through the bottom is given by

$$Q^* = 2\pi \int_0^{r_0} rQ(r)\,\mathrm{d}r = 0, \tag{2·37}$$

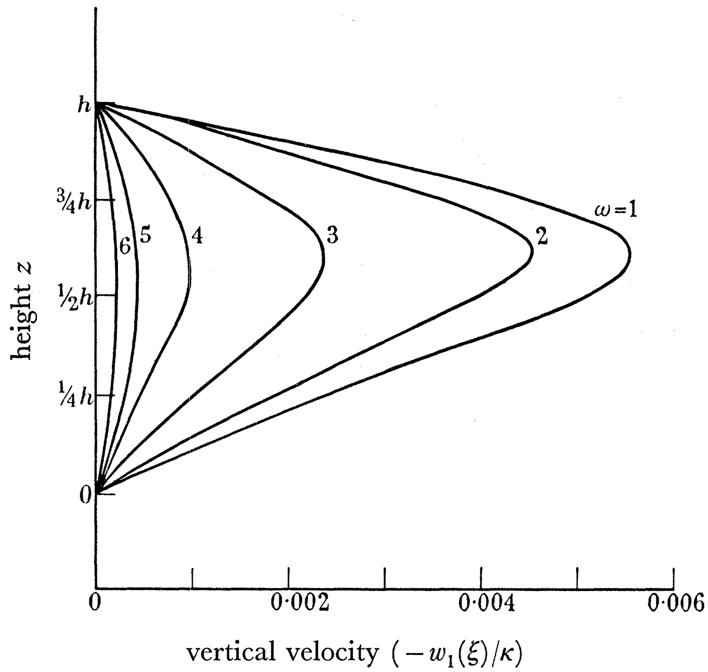

FIGURE 1. Vertical velocity profiles against height for values $\omega = 1, 2, 3, 4, 5, 6$.

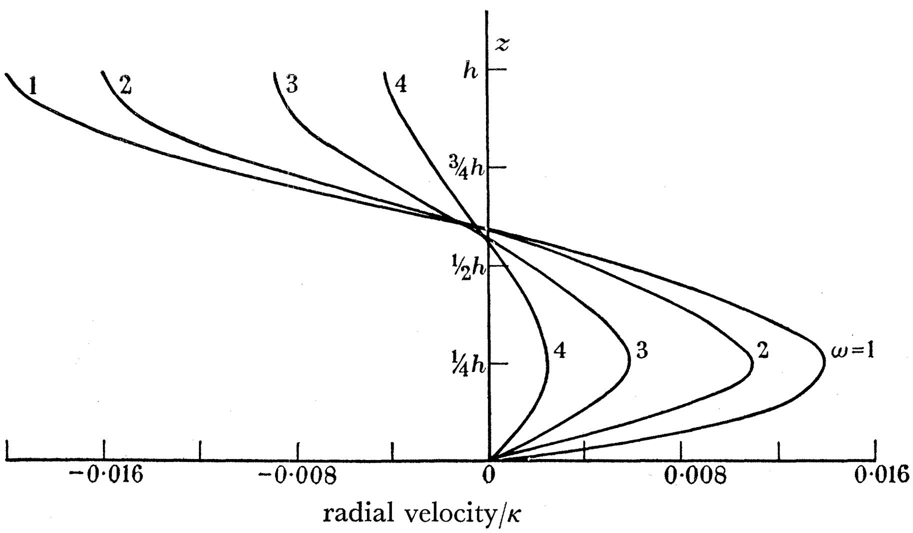

FIGURE 2. Radial velocity profiles against height for values $\omega = 1, 2, 3, 4$.

since $J_1(\beta_1 r_0) = J_1(\beta_2 r_0) = 0$. Hence as much heat is removed at the base as enters. With the above solutions (2·34), (2·35), (2·36), we can construct the general solution of the problem with an arbitrary heating distribution on the bottom.

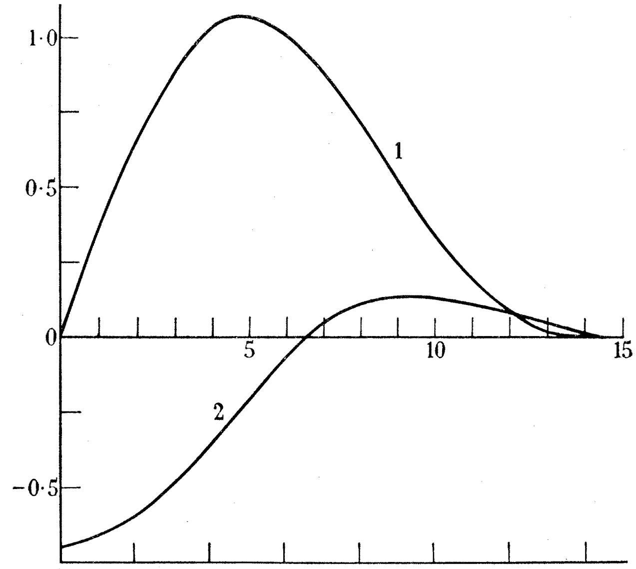

FIGURE 3. Velocity profiles in the radial direction when $w = 0$ at the side wall (not drawn to the same scale). Curve 1, zonal velocity, radial velocity; curve 2, vertical velocity.

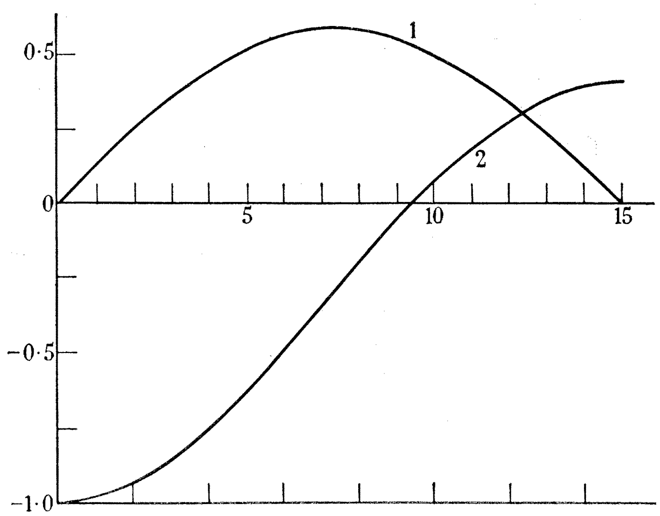

FIGURE 4. Velocity profiles in the radial direction when w is finite at the side wall (not drawn to the same scale). Curve 1, zonal velocity, radial velocity; curve, 2, vertical velocity.

In figures 1 to 4 it will be noted that the vertical velocity attains a maximum value at about $z = 0{\cdot}6h$, that the radial velocity attains a maximum value at $z = 0{\cdot}25h$, a zero at $z = 0{\cdot}6h$ and a minimum value on the free surface. These are the principal features of the vertical variation. The details of the radial variation of u and w may be observed in figures 1 to 4, where it will be noted that the maximum value of u occurs about 5 cm from the central axis. If we disregard the boundary condition $w = 0$ at $r = r_0$ and use only the condition $u = 0$ at $r = r_0$ the position of the maximum radial velocity is shifted to about 7·5 cm from the central axis. This indicates that the viscous conditions at the side of the vessel are unimportant in the present problem.

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3. Rotation Ω, with heating, steady symmetrical solutions. No non-linear terms

In this case we have the following equations governing the flow relative to the dishpan:

$$-2\Omega\rho_0 v = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u - \frac{u}{r^2}\right), \tag{3·1}$$

$$2\Omega\rho_0 u = \mu\left(\nabla^2 v - \frac{v}{r^2}\right), \tag{3·2}$$

$$0 = -\frac{\partial p}{\partial z} + g\alpha T + \mu\nabla^2 w, \tag{3·3}$$

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{3·4}$$

$$0 = Jk\nabla^2 T, \tag{3·5}$$

where ∇^2 is the same as in § 2. The normal types of solution can be obtained in the present case by taking

$$\left.\begin{aligned} u &= U_1(z)\,J_1(\beta r),\\ v &= V_1(z)\,J_1(\beta r),\\ w &= W_1(z)\,J_0(\beta r),\\ p &= P_1(z)\,J_0(\beta r),\\ \rho &= \rho_1(z)\,J_0(\beta r),\\ T &= T_1(z)\,J_0(\beta r),\end{aligned}\right\} \tag{3·6}$$

for with these substitutions we obtain the following ordinary system of equations, involving U_1, V_1, W_1, P_1 and T_1:

$$-2\Omega\rho_0 V_1 = \beta P_1 + \mu(U_1'' - \beta^2 U_1), \tag{3·7}$$

$$2\Omega\rho_0 U_1 = \mu(V_1'' - \beta^2 V_1), \tag{3·8}$$

$$-P_1' + g\alpha T_1 + \mu(W_1'' - \beta^2 W_1) = 0, \tag{3·9}$$

$$\beta U_1 + W_1' = 0, \tag{3·10}$$

$$T_1'' - \beta^2 T_1 = 0. \tag{3·11}$$

The boundary conditions at the top and bottom are now

$$\left.\begin{aligned} U_1 = V_1 = W_1 = 0, \quad T_1' = H, \quad z = 0;\\ U_1' = V_1' = W_1 = T_1' = 0, \quad z = h;\end{aligned}\right\} \tag{3·12}$$

corresponding to the same physical conditions as in § 2. We now make the transformation $z = h\xi$ and introduce the two parameters

$$a = \beta h, \quad R = \frac{\Omega\rho_0 h^2}{\mu}. \tag{3·13}$$

The first parameter is geometrical and has occurred in § 2, the second parameter is a rotation Reynolds number for the motion. In this case the above set of equations can be arranged conveniently in the form

$$(\mathrm{D}^2 - a^2)\,U_1 + 2R_1 V_1 + \frac{ah}{\mu} P_1 = 0, \tag{3·14}$$

$$-2R_1 U_1 + (\mathrm{D}^2 - a^2)\,V_1 = 0, \tag{3·15}$$

$$(\mathrm{D}^2 - a^2)\,W_1 - \frac{h}{\mu}\mathrm{D}P_1 = \frac{\kappa\cosh a(1-\xi)}{a\sinh a}, \tag{3·16}$$

$$aU_1 + \mathrm{d}W_1 = 0, \tag{3·17}$$

where κ is defined following (2·20) and where we have used (2·19). It is most convenient as in § 2 to derive the differential equation satisfied by W_1, and it is easily shown by elimination or otherwise that W_1 now satisfies the equation

$$(\mathrm{D}^2-a^2)^3 W_1+4R^2\mathrm{D}^2W_1 = 0. \tag{3·18}$$

This equation tends to (2·18) as $\Omega\to 0$ or $R\to 0$. It remains now to express the boundary conditions (3·12) in terms of W_1 and its derivates. It follows immediately that four of these conditions will be

$$\left.\begin{aligned} W_1(0) &= W_1'(0) = 0,\\ W_1(1) &= W_1''(1) = 0.\end{aligned}\right\} \tag{3·19}$$

The condition $V_1(0) = 0$ is a little more troublesome but leads ultimately to the condition

$$W_1^{\mathrm{v}}(0)-2a^2W_1'''(0) = a^2\kappa, \tag{3·19a}$$

while the condition $V_1'(1) = 0$ leads to

$$W_1^{\mathrm{iv}}(1) = -\frac{\kappa a}{\sinh a}. \tag{3·19b}$$

The six conditions (3·19), (3·19a), (3·19b) constitute the complete set of boundary conditions at the bottom of the fluid and at the free surface.

We can now proceed to the solution of (3·18) and its associated boundary conditions by the expansion method examined in § 2, and for this purpose we make the assumption that W_1 can be expanded in the form

$$W_1(\xi) = w_0(\xi)+a^2w_1(\xi)+a^4w_2(\xi)+\dots. \tag{3·20}$$

Interest centres principally on the leading term, since it is likely, but not proved here, that the succeeding terms will make small contributions to the ultimate value of $W_1(\xi)$. It then follows that $w_0(\xi)$ satisfies the following system:

$$\left.\begin{aligned} &w_0^{\mathrm{vi}}(\xi)+4R^2w_0''(\xi) = 0,\\ &w_0(0) = 0, \quad w_0(1) = 0,\\ &w_0'(0) = 0, \quad w_0''(1) = 0,\\ &w_0^{\mathrm{v}}(0) = 0, \quad w_0^{\mathrm{iv}}(1) = -\kappa.\end{aligned}\right\} \tag{3·21}$$

In the solution for $w_0(\xi)$ it is convenient to use a parameter ω in place of R which is defined by the equation

$$\omega = R^{\frac{1}{2}} = (\Omega\rho_0h^2/\mu)^{\frac{1}{2}}. \tag{3·22}$$

Without going into any detail it follows that the general solution for $w_0(\xi)$ may be written conveniently in the form

$$\begin{aligned} w_0(\xi) = A_0+A_1\xi+\sin\omega\xi[A_2\sinh\omega\xi+A_3\sinh\omega(1-\xi)]&\\ +\sin\omega(1-\xi)\,[A_4\sinh\omega\xi+A_5\sinh\omega(1-\xi)]&,\end{aligned}$$

and the six boundary conditions then uniquely determine the six constants A_5. The detail of the solution need not concern us, it is sufficient to state the result:

$$\begin{aligned}\frac{4\omega^4s_1S_1(S_1C_1-s_1c_1)\,w_0(\xi)}{-\kappa} = {}& s_1S_1(S_1C_1-s_1c_1)-(S_1C_1-s_1c_1)\sin\omega\xi\sinh\omega\xi\\ &+(s_1c_1C_1+s_1S_1^2-c_1S_1C_1)\sin\omega\xi\sinh\omega(1-\xi)\\ &+(s_1c_1C_1-s_1^2S_1-c_1S_1C_1)\sin\omega(1-\xi)\sinh\omega\xi\\ &-(S_1C_1-s_1c_1)\sin\omega(1-\xi)\sinh\omega(1-\xi),\end{aligned} \tag{3·23}$$

where for convenience we have used the following notation: $C_1 = \cosh\omega$, $S_1 = \sinh\omega$, $c_1 = \cos\omega$, $s_1 = \sin\omega$. It may be verified that (3·23) satisfies the system (3·21). We consider the solution (3·23) in some detail before determining the other velocity components. It is of interest first of all to deduce the form of $w_0(\xi)$ when Ω (or ω) is small, in order that we may judge the effect of a small rotation upon the flow. This involves a considerable amount of expansion of the various constituents of (3·23) and we quote just the final result:

$$\frac{w_0(\xi)}{-\kappa} = \tfrac{1}{48}\xi^2(1-\xi)\left\{(3-2\xi)+\frac{\omega^4}{420}[-19+6\xi+6\xi^2+6\xi^3-8\xi^4+2\xi^5]+\dots\right\}. \quad (3·24)$$

It will be observed that as $\Omega \to 0$ the expression for $w_0(\xi)$ given here tends to that in the non-rotating case (2·30). The second term within the brackets is always negative when $0 \leqslant \xi \leqslant 1$, hence the first effect of a small rotation upon the vertical velocity is to decrease it at all points. This decrease of $w_0(\xi)$ with increasing rotation is generally true as we increase Ω from zero to infinity (see figure 1). It is easily shown that when the rotation Ω becomes sufficiently large the asymptotic form of $w_0(\xi)$ at all points $0<\xi<1$ will be given by making ω_1 large in (3·23). This leads to the result

$$\frac{w_0(\xi)}{-\kappa} \sim \frac{1}{4\omega^4}. \quad (3·25)$$

This result indicates that apart from the boundaries, where the vertical velocity is zero, the vertical velocity tends to assume a constant value. In this connexion it is of interest to note that for small values of ξ, that is, near the bottom, the expression for the vertical velocity from (3·23) becomes

$$\frac{w_0(\xi)}{-\kappa} = \frac{\xi^2}{4\omega^2}. \quad (3·26)$$

Hence the value of $-1/4\omega^4$ which obtains in the free stream will be attained in a vertical distance δ given by $\delta^2 = \omega^{-2}$, so that

$$\delta = \omega^{-1} = R^{-\frac{1}{4}}. \quad (3·27)$$

This thickness δ is of course the boundary layer at the bottom, which decreases with increasing Reynolds number. It may be mentioned here that this boundary-layer result holds also for the vertical velocity at the free surface, that is, the free-stream vertical velocity decreases to zero in a layer of thickness δ at the free surface.

We proceed now to the radial component of velocity $U_1(z)$. This must be expanded in the form

$$U_1 = \frac{1}{a}u_0(\xi) + au_1(\xi) + a^3u_2(\xi) + \dots, \quad (3·28)$$

and the leading term of this development is given by

$$u_0(\xi) = -w_0'(\xi). \quad (3·29)$$

Hence we obtain

$$\begin{aligned}\frac{4\omega^3 s_1 S_1(S_1C_1 - s_1c_1)\,u_0(\xi)}{\kappa} = &-(S_1C_1-s_1c_1)\{\sin\omega\xi\cosh\omega\xi+\cos\omega\xi\sinh\omega\xi\}\\ &+(s_1c_1C_1-s_1S_1^2-c_1S_1C_1)\{-\sin\omega\xi\cosh\omega(1-\xi)+\cos\omega\xi\sinh\omega(1-\xi)\}\\ &+(s_1c_1C_1-s_1^2S_1-c_1S_1C_1)\{\sin\omega(1-\xi)\cosh\omega\xi-\cos\omega(1-\xi)\sinh\omega\xi\}\\ &+(S_1C_1-s_1c_1)\{\cos\omega(1-\xi)\sinh\omega(1-\xi)+\sin\omega(1-\xi)\cosh\omega(1-\xi)\}.\end{aligned} \quad (3·30)$$

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When Ω increases from zero the successive profiles of $u_0(\xi)$ are indicated in figure 2. It will be observed that $u_0(\xi)$ decreases rapidly to zero with increasing ω, and when we investigate the asymptotic form of $u_0(\xi)$ for large ω we find that $u_0(\xi) \sim e^{-\omega\xi}$ in $0<\xi<1$. Thus the radial component of velocity decreases exponentially with ω compared with the algebraic decrease of $w_0(\xi)$ (see equation (3·25)). We may quote two results of interest. At the free surface for small ω_1 we have

$$\frac{u_0(1)}{\kappa} = \frac{1}{48}\left\{-1+\frac{\omega^4}{60}+\dots\right\}, \tag{3·31}$$

which indicates the rapidity of decrease of $u_0(1)$ with increasing ω_1. At the bottom we have the approximate formula

$$\frac{u_0(\xi)}{\kappa} = -\tfrac{1}{2}\xi\frac{(C_1-c_1)\,(S_1-s_1)}{(S_1C_1-s_1c_1)\,\omega^2}. \tag{3·32}$$

Consider next the zonal velocity. This may be expanded in ascending powers of a in exactly the same form as u, and we write

$$V_1(\xi) = \frac{1}{a}v_0(\xi)+av_1(\xi)+a^3v_2(\xi)+\dots. \tag{3·33}$$

It follows from (3·15) and (3·17) that

$$v_0''(\xi) = -2Rw_0'(\xi),$$

hence

$$v_0'(\xi) = -2Rw_0(\xi)$$

(there is no arbitrary constant here since $v_0'(1) = 0$, $w_0(1) = 0$). The final formula for the zonal velocity is therefore

$$v_0(\xi) = -2R\int_0^\xi w_0(\xi)\,d\xi, \tag{3·34}$$

and we then obtain

$$\begin{aligned}\frac{2\omega^2s_1S_1(S_1C_1-s_1c_1)\,v_0(\xi)}{\kappa} &= s_1S_1(S_1C_1-s_1c_1)\,\xi-\frac{1}{2\omega_1}(S_1C_1-s_1c_1)\,\{\sin\omega\xi\cosh\omega\xi-\cos\omega\xi\sinh\omega\xi\}\\ &+\frac{1}{2\omega}(s_1c_1C_1-s_1S_1^2-c_1S_1C_1)\,\{S_1-\sin\omega\xi\cosh\omega(1-\xi)-\cos\omega\xi\sinh\omega(1-\xi)\}\\ &+\frac{1}{2\omega}(s_1c_1C_1-s_1^2S_1-c_1S_1C_1)\,\{-s_1+\sin\omega(1-\xi)\cosh\omega\xi+\cos\omega(1-\xi)\sinh\omega\xi\}\\ &-\frac{1}{2\omega}(S_1C_1-s_1c_1)\,\{s_1C_1-c_1S_1-\sin\omega(1-\xi)\cosh\omega(1-\xi)+\cos\omega(1-\xi)\sinh\omega(1-\xi)\}.\end{aligned} \tag{3·35}$$

The expansion for $v_0(\xi)$ which is valid for small rotation speeds is given by

$$\frac{v_0(\xi)}{\kappa} = \frac{\omega^2}{24}\left\{(\xi^3-\tfrac{5}{4}\xi^4+\tfrac{2}{5}\xi^5)+\frac{\omega^4}{420}(-\tfrac{19}{3}\xi^3+\tfrac{25}{4}\xi^4-2\xi^7+\tfrac{5}{4}\xi^8-\tfrac{2}{9}\xi^9)+\dots\right\}. \tag{3·36}$$

This formula shows that $v_0(\xi)\to 0$ for all ξ as $\Omega\to 0$, and that near the bottom $v_0(\xi) \propto \xi^3$, which indicates a very slow increase of zonal velocity with increasing height above the bottom. On the free surface with small values of ω we have

$$\frac{v_0(1)}{\kappa} = \frac{\omega^2}{160}\left\{1-\frac{19\omega^4}{1134}\dots\right\}. \tag{3·37}$$

This suggests that $v_0(1)$ will probably increase from zero to some maximum value with increasing ω. This is, indeed, true, for when we use (3·35) and plot $v_0(1)$ against ω the curve (figure 5) shows that $v_0(1)$ attains a maximum around $\omega = 3$ and thereafter falls off to zero. The corresponding curve of $v_0(\frac{1}{2})$ against ω is included in the same diagram in order to indicate the relative magnitudes of the zonal velocity at $z = \frac{1}{2}h$ and $z = h$. For large values of the Reynolds number the asymptotic formula for $v_0(\xi)$ is given by

$$\frac{v_0(\xi)}{\kappa} \sim \frac{\xi - 1/\omega}{2\omega_1^2} \quad (0<\xi<1), \tag{3·38}$$

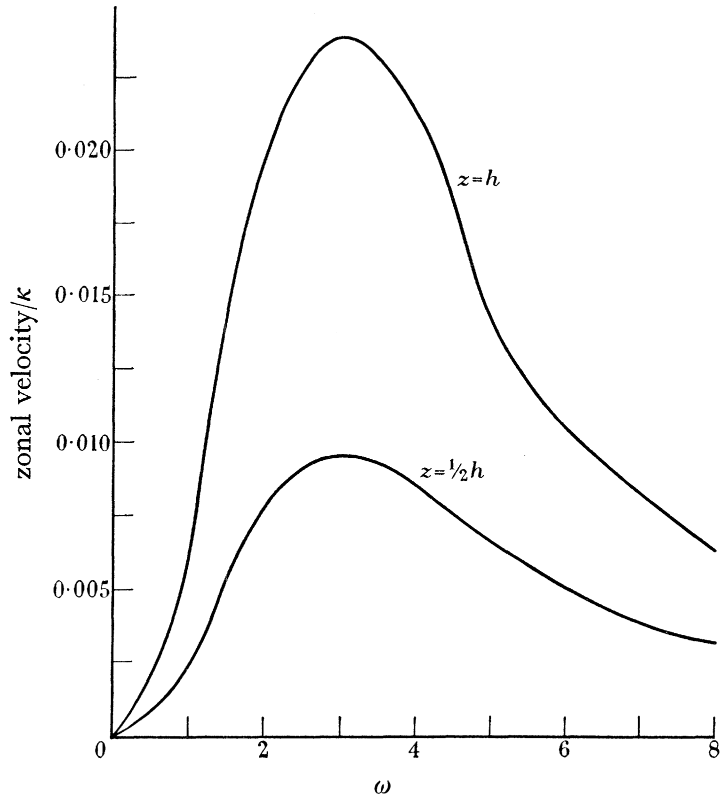

FIGURE 5. Zonal velocity against ω at $z=h$ and $z=\frac{1}{2}h$.

hence the flow tends to become a uniform shearing flow with its maximum at the free surface. Here also, however, $v_0(\xi) \to 0$ as $\omega \to \infty$. Thus all three velocity components tend to zero as the rotation increases to fairly high values, and it would appear therefore that when Ω is sufficiently great the fluid is essentially in solid rotation. This suggests that the energy which is derived from heating may be more readily available for asymmetrical flows as the rotation rate increases.†

It may be noted also that at the free surface the ratio of radial velocity to zonal velocity is given by

$$\tan\beta_0 = -\frac{u_0(1)}{v_0(1)} = \frac{10}{3R}\frac{1-\frac{1}{60}\omega^4\ldots}{1-\frac{19}{1134}\omega^4\ldots}, \tag{3·39}$$

provided the Reynolds number is sufficiently small. When ω is moderately large we have

$$\frac{u_0(1)}{\kappa} \sim -\frac{1}{4\omega^3}, \tag{3·40}$$

hence

$$\frac{u_0(1)}{v_0(1)} \sim -\frac{1}{2\omega}. \tag{3·41}$$

† Alternatively, the mechanism of the symmetrical flow may be different.

From these results (3·39) and (3·41) we can determine how quickly the flow at the free surface becomes 'geostrophic'. There are no longitudinal variations of the pressure p in the present problem, hence the isobars are concentric circles. The angle β_0 in (3·39) is the angle of the resultant velocity vector to such a circle (more precisely with the positively drawn tangent). We note that when $R = 3\frac{1}{3}$, then β_0 is approximately 45°. When $R = 42$, $\omega = 6{\cdot}5$ approximately, β_0 is approximately $4\frac{1}{2}$°—this corresponds to the low-rotation régime. Within the fluid, since the radial velocity decreases exponentially, it follows that geostrophic flow will be attained more quickly than on the free surface.

There is no necessity to take this velocity investigation further. The satisfying of the condition $w = 0$ at $r = r_0$ will proceed exactly as in § 2, and the radial profile of v will be precisely the same as the radial profile of u. Thus the maximum value of the zonal velocity will occur at about 5 cm from the central axis (with $w = 0$ at $r = 15$) and at 7·5 cm from the central axis with w non-zero at the side. It has been pointed out recently during a discussion at the University of Chicago that the term v^2/r becomes of similar order of magnitude to the Coriolis term in the low-rotation experiments. This will doubtless influence the quantitative aspects of the present investigation but is unlikely to affect the qualitative picture which has been presented here. It will be possible to obtain some insight into the effect of this particular non-linear term in § 6.

It will be noted that the result (3·38) combines with (3·6) to give

$$v = v^* \sim \frac{\kappa}{2Ra}\left(\xi - \frac{1}{\omega}\right) J_1(\beta r), \tag{3·42}$$

when the Reynolds number R is sufficiently high. This result may be deduced simply as follows. From a consideration of the orders of magnitude of the vertical, radial and zonal velocities, the pressure and temperature fields as functions of the Reynolds number, it follows that the equations (3·1) to (3·5) reduce to the following set when R is sufficiently large:

$$2\Omega\rho_0 v = \frac{\partial p}{\partial r}, \tag{3·43}$$

$$g\alpha T = \frac{\partial p}{\partial z}, \tag{3·44}$$

$$\nabla^2 T = 0. \tag{3·45}$$

When the $1/\omega$ term of (3·42) is omitted the velocity v^* and the temperature

$$T = -hH \cosh a(1-\xi)\, J_0(\beta r)/a \sinh a,$$

with a small, represent solutions of (3·43) to (3·45). Such a v^* is known in meteorology as the thermal wind.

4. The effect of the inertia term $w\dfrac{\partial v}{\partial z}$

It has been shown in § 3 that when the rotation Reynold's number R becomes sufficiently large ($R > 15$ say) that the flow tends to become purely zonal. R cannot be made indefinitely large, however, because in some parts of the field the non-linear inertia terms then assume an increasing importance, namely, in those parts where the boundary-layer effects are sensible. These effects are present of course at the very lowest Reynolds numbers but may be ignored then because of their smallness; we may say that the results of § 3 are adequate in the range $0<\omega<3{\cdot}5$ provided H is sufficiently small.

Since the inertia terms are non-linear and their influence is difficult to assess, it is proposed to investigate three of the non-linear terms only and to investigate them separately. In general, the effects of the non-linear terms will not be additive, but in the present approximate method of dealing with these terms it so happens that we may add together the effects of these three inertia terms, and nothing is lost, therefore, by the above separation. In the present section we investigate the effect of the term $w(\partial v/\partial z)$ which appears on the left-hand side of the zonal equation of motion. It is of significance that in this term the factor $\partial v/\partial z$ quickly attains its asymptotic value $\partial v^*/\partial z$ when $R>15$. We adopt an iterative procedure and shall approximate to this term by writing it in the form $w(\partial v^*/\partial z)$, where v^*, defined in (3·42), is the zonal velocity which develops as R becomes sufficiently large. It is not actually necessary to perform this substitution, for the method we use could equally apply to the term in its original form, but the detailed working is made rather easier by the substitution of v^* for v and the result to the present approximation is not affected. Since we have replaced v by its asymptotic value for large R we can ultimately study the solution of the equations only in the large R range. In order to study this term in the small R range we should have to replace v by the expression (3·36), but since interest centres principally in the range of large R all this section will be devoted to this range. The approximation made above is not sufficient to make the equations tractable, and I shall approximate further in the equations of motion by taking $\nabla^2 \equiv \partial^2/\partial z^2$ in the Navier-Stokes terms, that is, the differentials with respect to r are ignored. This may be justified because of the shallowness of the liquid contained in the dishpan and is in fact a similar approximation to that made in earlier sections where the velocity components are expanded in powers of a. A similar approximation is permissible in dealing with atmospheric flow on the spherical earth. The equations governing the flow are taken to be

$$-2\Omega\rho_0 v = -\frac{\partial p}{\partial r} + \mu\frac{\partial^2 u}{\partial z^2}, \tag{4·1}$$

$$\rho_0 w\frac{\partial v^*}{\partial z} + 2\Omega\rho_0 u = \mu\frac{\partial^2 v}{\partial z^2}, \tag{4·2}$$

$$0 = -\frac{\partial p}{\partial z} + g\alpha T, \tag{4·3}$$

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{4·4}$$

$$\nabla^2 T = 0. \tag{4·5}$$

It will be noted also that in equation (4·3) the term $\mu(\partial^2 w/\partial z^2)$ is omitted; evidently this does not prevent us satisfying the boundary conditions at the surfaces $z=0$ and $z=h$, and it is the satisfying only of the side condition $w=0$ at $r=a$ that is invalidated by this step. This side condition has been ignored also in § 3, and it will be shown later that the solution of the above equations is similar to that of the previous section when the term $w(\partial v/\partial z)$ tends to zero.

With the conditions (2·15) imposed upon T we are led to

$$T = -\frac{hH\cosh a(1-\xi)}{a\sinh a} J_0(\beta r). \tag{4·6}$$

From (3·42) we have

$$\frac{\partial v^*}{\partial z} = \frac{\kappa}{2h\omega^2 a} J_1(\beta r). \tag{4·7}$$

If we eliminate the pressure between (4·1) and (4·3) we obtain

$$-2\Omega\rho_0\frac{\partial v}{\partial z} = -g\alpha\frac{\partial T}{\partial r} + \mu\frac{\partial^3 u}{\partial z^3}. \tag{4·8}$$

Eliminating v between (4·2) and (4·8) leads to

$$\frac{\partial^4 u}{\partial z^4} + \frac{4\Omega^2\rho_0^2}{\mu^2}u + \frac{2\Omega\rho_0^2}{\mu^2}w\frac{\partial v^*}{\partial z} = \frac{g\alpha}{\mu}\frac{\partial^2 T}{\partial z\,\partial r}.$$

If in this equation we replace z by $h\xi$ as in § 2 and replace $\partial v^*/\partial z$ using (4·7) we obtain

$$\frac{\partial^4 u}{\partial \xi^4} + \frac{4\Omega^2\rho_0^2 h^4}{\mu^2}u + \frac{\rho_0\kappa h}{a\mu}wJ_1(\beta r) = -\frac{Hg\alpha\beta h^4}{\mu}\frac{\sinh a(1-\xi)}{\sinh a}J_1(\beta r).$$

We now introduce a dimensionless Stokes stream function ψ which is such that

$$ru = -\kappa h^2\frac{\partial\psi}{\partial z}, \quad rw = \kappa h^2\frac{\partial\psi}{\partial r}, \tag{4·9}$$

and the equation for ψ is then

$$\frac{\partial^5\psi}{\partial\xi^5} + 4\omega^4\frac{\partial\psi}{\partial\xi} - \epsilon J_1(\eta)\frac{\partial\psi}{\partial\eta} = \frac{\sinh a(1-\xi)}{\sinh a}\eta J_1(\eta), \tag{4·10}$$

where ω is defined in (3·22) and where

$$\eta = \beta r, \quad \epsilon = \frac{\kappa h\rho_0}{\mu}, \quad \epsilon^* = \frac{\epsilon}{4\omega^4}. \tag{4·11}$$

The term on the right-hand side of (4·10) arises from the temperature field. The quantity ϵ in (4·11) is a non-dimensional constant which, through κ, contains the constant H in the numerator. Since H, which is related to the temperature difference between the central axis and the rim of the dishpan, can be made as small as we please, it follows that ϵ can be made as small as we please. It appears, however, that the experimental value of ϵ is of the same order of magnitude as $4\omega^4$, and thus the results obtained here may not give good quantitative agreement with the actual experiment but will serve as an adequate guide to the qualitative effect of the inertia term. The term which involves ϵ in (4·10) arises from the term $w(\partial v^*/\partial z)$ in (4·2), and the above analysis shows that this term can be neglected in the problem only when the difference in temperature between the rim and centre of the dishpan is sufficiently small. Since we wish to investigate the influence of this term upon the flow of § 3, it is convenient to look for a solution of (4·10) which will be of the form

$$\psi = \psi_0 + \epsilon^*\psi_1 + \epsilon^{*2}\psi_2 + \dots, \tag{4·12}$$

where the ψ_s's are independent of ϵ^*. It may then be anticipated that ψ_0 represents essentially the flow of § 3 while $\psi_1, \psi_2, \dots$ will represent the departure from the ψ_0 flow due to the term $w(\partial v^*/\partial z)$. Since ϵ^* can be made as small as we please the convergence of (4·12) can always be arranged.

When we substitute (4·12) in (4·10) and equate to zero successive powers of ϵ^* we obtain

$$\frac{\partial^5\psi_0}{\partial\xi^5}+4\omega^4\frac{\partial\psi_0}{\partial\xi}=\frac{\sinh a(1-\xi)}{\sinh a}\eta J_1(\eta), \tag{4·13}$$

$$\frac{\partial^5\psi_1}{\partial\xi^5}+4\omega^4\frac{\partial\psi_1}{\partial\xi}=4\omega^4 J_1(\eta)\frac{\partial\psi_0}{\partial\eta}, \tag{4·14}$$

and so on. We limit the present investigation to the determination of ψ_0 and ψ_1, but if desired higher approximations can easily be obtained. We consider first the function ψ_0. From (3·12) ψ_0 has to satisfy the conditions

$$\left.\begin{aligned}\psi_0&=0, \quad \xi=0;\\ \psi_0&=0, \quad \xi=1;\\ \frac{\partial\psi_0}{\partial\xi}&=0, \quad \xi=0;\\ \frac{\partial^2\psi_0}{\partial\xi^2}&=0, \quad \xi=1;\end{aligned}\right\} \tag{4·15}$$

and further restrictions will be imposed later. The complete solution of (4·13) will be

$$\psi_0=-\frac{\cosh a(1-\xi)}{a(4\omega^4+a^4)\sinh a}\eta J_1(\eta)+F_0+A\sin\omega\xi\sinh\omega\xi+B\sinh\omega\xi\sin\omega(1-\xi)$$
$$+C\sinh\omega(1-\xi)\sin\omega\xi+D\sinh\omega(1-\xi)\sin\omega(1-\xi), \tag{4·16}$$

where F_0, A, B, C and D are functions of η only. Using the same notation as in (3·23) it follows from (4·15) that

$$\left.\begin{aligned}F_0+DS_1s_1&=\frac{\cosh a}{a(4\omega^4+a^4)\sinh a}\eta J_1(\eta),\\ F_0+AS_1s_1&=\frac{1}{a(4\omega^4+a^4)\sinh a}\eta J_1(\eta),\\ Bs_1+CS_1-D(C_1s_1+S_1c_1)&=\frac{-1}{\omega(4\omega^4+a^4)}\eta J_1(\eta),\\ AC_1c_1-BC_1-Cc_1+D&=\frac{a}{2\omega^2(4\omega^4+a^4)\sinh a}\eta J_1(\eta).\end{aligned}\right\} \tag{4·17}$$

We can solve these equations for A, B, C and D and thus write (4·16) in the form

$$\psi_0=\frac{\eta J_1(\eta)}{a(4\omega^4+a^4)\sinh a}\{-\cosh a(1-\xi)+\alpha(\xi)\}+F_0(\eta)\,G_0(\xi), \tag{4·18}$$

where

$$G_0(\xi)=1-\frac{\sin\omega\xi\sinh\omega\xi}{s_1S_1}-\frac{\sin\omega(1-\xi)\sinh\omega(1-\xi)}{s_1S_1}$$
$$-\frac{S_1^2s_1+S_1C_1c_1-s_1c_1C_1}{S_1s_1(S_1C_1-s_1c_1)}\sin\omega\xi\sinh\omega(1-\xi)-\frac{s_1^2S_1+c_1S_1C_1-s_1c_1C_1}{S_1s_1(S_1C_1-s_1c_1)}\sin\omega(1-\xi)\sinh\omega\xi, \tag{4·19}$$

and

$$\alpha(\xi)=A'\sin\omega\xi\sinh\omega\xi+B'\sinh\omega\xi\sin\omega(1-\xi)+C'\sinh\omega(1-\xi)\sin\omega\xi$$
$$+D'\sinh\omega(1-\xi)\sin\omega(1-\xi),$$

with $\alpha(0) = \cosh a$, $\alpha(1) = 1$, $\alpha'(0) = a\sinh a$, $\alpha''(1) = a^2$. In § 3 the velocity field was obtained in ascending powers of a, and if we continue with that procedure here it will be noted that

$$\alpha(\xi) = 1 - G_0(\xi) + O(a^2).$$

Hence we obtain from (4·18)

$$\psi_0 = \frac{\eta J_1(\eta)}{4\omega^4 a^2}\{1 - G_0(\xi) - \cosh a(1-\xi)\} + F_0(\eta)\, G_0(\xi),$$

that is, $$\psi_0 = -\frac{1}{4\omega^4 a^2}\eta J_1(\eta)\, G_0(\xi) + F_0(\eta)\, G_0(\xi) + \text{higher powers of } a. \tag{4·20}$$

By comparing (4·19) and (3·23) it will be noted that

$$4\omega^4 w_0(\xi) = -\kappa G_0(\xi). \tag{4·21}$$

Consider now the function ψ_1. We have from (4·14) and (4·20)

$$\frac{\partial^5\psi_1}{\partial\xi^5} + 4\omega^4\frac{\partial\psi_1}{\partial\xi} = 4\omega^4 G_0(\xi)\left\{-\frac{1}{4\omega^4 a^2}\eta J_0(\eta)\, J_1(\eta) + J_1(\eta)\frac{\mathrm{d}F_0}{\mathrm{d}\eta}\right\}, \tag{4·22}$$

and since the particular solution of the differential equation

$$\frac{\partial^5\psi_1}{\partial\xi^5} + 4\omega^4\frac{\partial\psi_1}{\partial\xi} = G_0(\xi)$$

is $$\psi_1 = \xi(5 - G_0)/16\omega^4,$$

it follows that the complete solution of (4·22) will be

$$\psi_1 = \tfrac{1}{4}\xi(5 - G_0)\left\{-\frac{1}{4\omega^4 a^2}\eta J_0(\eta)\, J_1(\eta) + J_1(\eta)\frac{\mathrm{d}F_0}{\mathrm{d}\eta}\right\} + \Psi_1, \tag{4·23}$$

where

$$\Psi_1 = F_1 + \alpha_1\sinh\omega\xi\sin\omega\xi + \beta_1\sinh\omega\xi\sin\omega(1-\xi) + \gamma_1\sinh\omega(1-\xi)\sin\omega\xi + \delta_1\sinh\omega(1-\xi)\sin\omega(1-\xi), \tag{4·24}$$

F_1, α_1, β_1, γ_1 and δ_1 being functions of η only. The function ψ_1 has to satisfy the same boundary conditions as ψ_0, and hence we have the following four relations between these five functions:

$$\left.\begin{aligned} F_1 + \delta_1 S_1 s_1 &= 0, \\ F_1 + \alpha_1 S_1 s_1 &= -\frac{5}{4}\left\{-\frac{1}{4\omega^4 a^2}\eta J_0(\eta)\, J_1(\eta) + J_1(\eta)\frac{\mathrm{d}F_0}{\mathrm{d}\eta}\right\}, \\ \beta_1 s_1 + \gamma_1 S_1 - \delta_1(C_1 s_1 + S_1 c_1) &= -\frac{5}{4\omega}\left\{-\frac{1}{4\omega^4 a^2}\eta J_0(\eta)\, J_1(\eta) + J_1(\eta)\frac{\mathrm{d}F_0}{\mathrm{d}\eta}\right\}, \\ \alpha_1 C_1 c_1 - \beta_1 C_1 - \gamma_1 c_1 + \delta_1 &= \frac{G_0'(1)}{4\omega^2}\left\{-\frac{1}{4\omega^4 a^2}\eta J_0(\eta)\, J_1(\eta) + J_1(\eta)\frac{\mathrm{d}F_0}{\mathrm{d}\eta}\right\}. \end{aligned}\right\} \tag{4·25}$$

These equations we can solve for α_1, β_1, γ_1 and δ_1 in terms of F_1, F_0 and the various Bessel functions. Before proceeding any further with ψ_1 and this set of equations we consider the zonal velocity v, since in satisfying the boundary conditions upon v, namely,

$$v = 0, \quad \xi = 0; \qquad \frac{\partial v}{\partial\xi} = 0, \quad \xi = 1, \tag{4·26}$$

we thereby define the functions F_0 and F_1. From (4·8) we have

$$2\Omega\rho_0\frac{\partial v}{\partial\xi}=\frac{Hg\alpha h^2\beta}{a\sinh a}\cosh a(1-\xi)\,J_1(\eta)+\frac{\kappa\mu h^2}{rh^3}\frac{\partial^4\psi}{\partial\xi^4},$$

and if we integrate this with respect to ξ we have

$$2\Omega\rho_0 v=-\frac{Hg\alpha h^2\beta}{a^2\sinh a}\sinh a(1-\xi)\,J_1(\eta)+\frac{\kappa\mu}{rh}\frac{\partial^3\psi}{\partial\xi^3}+2\Omega\rho_0\chi(r),$$

where χ is an arbitrary function of r. Introducing the constants κ and ω we can write this expression for v in the form

$$v=-\frac{\kappa}{2\omega^2 a\sinh a}\sinh a(1-\xi)\,J_1(\eta)+\frac{\kappa h}{2r\omega^2}\frac{\partial^3\psi}{\partial\xi^3}+\chi. \tag{4·27}$$

In order to satisfy the first condition in (4·26) we choose χ so that

$$\chi(r)=\frac{\kappa}{2\omega^2 a}J_1(\eta)-\frac{\kappa h}{2r\omega^2}\left(\frac{\partial^3\psi}{\partial\xi^3}\right)_{\xi=0}. \tag{4·28}$$

Since
$$\frac{\partial v}{\partial\xi}=\frac{\kappa}{2\omega^2 a}\cosh a(1-\xi)\,J_1(\eta)+\frac{\kappa h}{2r\omega^2}\left\{\frac{\partial^4\psi_0}{\partial\xi^4}+\epsilon^*\frac{\partial^4\psi_1}{\partial\xi^4}+\dots\right\}, \tag{4·29}$$

it follows that we can satisfy the second condition in (4·26) by choosing F_0 and F_1 so that

$$\frac{1}{2\omega^2 a}J_1(\eta)+\frac{h}{2r\omega^2}\left(\frac{\partial^4\psi_0}{\partial\xi^4}\right)_{\xi=1}=0, \tag{4·30}$$

$$\left(\frac{\partial^4\psi_1}{\partial\xi^4}\right)_{\xi=1}=0. \tag{4·31}$$

Evidently this process can be repeated if higher-order terms ψ_s are required in (4·12). Using (4·20) and the result $G_0^{\text{iv}}(1)=4\omega^4$ it follows that

$$\frac{\partial^4\psi_0}{\partial\xi^4}=4\omega^4F_0(\eta)-\frac{1}{a^2}\eta J_1(\eta),$$

and thus, bearing in mind the definitions $a=\beta h$, $\eta=\beta r$, it follows from (4·30) that

$$F_0(\eta)\equiv 0. \tag{4·32}$$

Thus we have, as in § 3,
$$\psi_0=-\frac{1}{4\omega^4a^2}\eta J_1(\eta)\,G_0(\xi), \tag{4·33}$$

and the equations (4·25) simplify by the vanishing of the last term on the right-hand side of each equation. We now solve the equations (4·25). It is evident that the terms F_1 and $\eta J_0(\eta)\,J_1(\eta)$ can be dealt with separately. Corresponding to the former the solution for Ψ_1 is $\Psi_1'=G_0(\xi)\,F_1(\eta)$, and corresponding to the latter the solution is

$$\begin{aligned}\Psi_1''=\frac{5\eta J_0J_1}{16\omega^4a^2}\Bigg\{&\frac{\sinh\omega\xi\sin\omega\xi}{S_1s_1}+\frac{\dfrac{C_1c_1}{s_1}+\dfrac{S_1G_0'(1)}{5\omega^2}-\dfrac{c_1}{\omega}}{S_1C_1-s_1c_1}\sinh\omega\xi\sin\omega(1-\xi)\\&+\frac{\dfrac{C_1}{\omega}-\dfrac{C_1c_1}{S_1}-\dfrac{s_1G_0'(1)}{5\omega^2}}{S_1C_1-s_1c_1}\sinh\omega(1-\xi)\sin\omega\xi\Bigg\}\\=\frac{5\eta J_0J_1}{16\omega^4a^2}&G_1(\xi).\end{aligned} \tag{4·34}$$

Accordingly, the complete solution for Ψ_1 will be

$$\Psi_1 = G_0(\xi)\,F_1(\eta)+\frac{5}{16\omega^4a^2}\eta J_0 J_1 G_1(\xi). \tag{4·35}$$

From (4·23) and (4·31) we have

$$\frac{1}{4}\left[\frac{d^4}{d\xi^4}\{\xi(5-G_0)\}\right]_{\xi=1}\left\{-\frac{1}{4\omega^4a^2}\eta J_0 J_1\right\}+G_0^{iv}(1)\,F_1(\eta)+\frac{5}{16\omega^4a^2}\eta J_0 J_1 G_1^{iv}(1)=0,$$

that is, $$\frac{1}{16\omega^4a^2}\eta J_0 J_1\{G_0^{iv}(1)+4G_0'''(1)+5G_1^{iv}(1)\}+G_0^{iv}(1)F_1(\eta)=0.$$

The function $G_1(\xi)$ defined in (4·34) satisfies the differential equation $G_1^{iv}(\xi)+4\omega^4G_1(\xi)=0$, and thus since $G_1(1)=1$ we have $G_1^{iv}(1)=-4\omega^4$. Combining this with $G_0^{iv}(1)=4\omega^4$ we have

$$F_1(\eta)+\frac{1}{16\omega^4a^2}\eta J_0 J_1\left\{1+\frac{G_0'''(1)}{\omega^4}-5\right\}=0,$$

and hence $$F_1(\eta)=\frac{1}{16\omega^4a^2}\eta J_0 J_1\left\{4-\frac{G_0'''(1)}{\omega^4}\right\}. \tag{4·36}$$

The function ψ_1 is thus given by

$$\psi_1=\frac{1}{16\omega^4a^2}\eta J_0 J_1\left\{-\xi(5-G_0)+G_0\left(4-\frac{G_0'''(1)}{\omega^4}\right)+5G_1\right\}. \tag{4·37}$$

It will be noted that ψ_1 vanishes at $r=0$ and also at $r=a$ provided $J_1(\beta a)=0$ as in previous sections. Thus the central axis of the dishpan and the side are streamlines as required. The function ψ_1, unlike ψ_0, has a zero at the first vanishing point of $J_0(\beta r)$, and this fact has interesting consequences which will be mentioned later. In (4·37) it is easily shown that

$$G_0'(1)=-\frac{\omega(S_1-s_1)^2}{S_1C_1-s_1c_1},\quad G_0'''(1)=\frac{2\omega^3(C_1-c_1)^2}{S_1C_1-s_1c_1}. \tag{4·38}$$

The formula for v in (4·27) can be expressed now entirely in terms of the $G_s(\xi)$ and Bessel functions. The complete formula for ψ is

$$\psi=-\frac{1}{4\omega^4a^2}\eta J_1 G_0+\frac{\epsilon^*}{16\omega^4a^2}\eta J_0 J_1\left\{-\xi(5-G_0)+G_0\left(4-\frac{G_0'''(1)}{\omega^4}\right)+5G_1\right\}+O(\epsilon^{*2}), \tag{4·39}$$

and the complete expression for v is

$$v=\frac{\kappa}{2\omega^2a}\xi J_1(\eta)+\frac{\kappa J_1(\eta)}{8\omega^6a}\left\{G_0'''(0)-G_0'''(\xi)+\frac{\epsilon^*J_0}{4}\right.$$
$$\left.\times\left[\xi G_0'''(\xi)+3G_0''(\xi)-3G_0''(0)+5G_1'''(\xi)-5G_1'''(0)+\left(4-\frac{G_0'''(1)}{\omega^4}\right)(G_0''(\xi)-G_0''(0))\right]+O(\epsilon^{*2})\right\}. \tag{4·40}$$

In (4·40) the function $\sinh a(1-\xi)$ has been expanded in powers of a. In order to get a clearer idea of the behaviour of these solutions it is necessary to investigate $G_0(\xi)$ and $G_1(\xi)$ a little more closely. As was stated earlier the solutions obtained in this section will be valid for large R only, that is, $\omega > 3·5$ say. We can investigate therefore the asymptotic behaviour of G_0 and G_1 for large ω. Here there are two cases to be considered:

(*a*) large ω, $0<\xi<1$, large $\omega\xi$, i.e. outside the boundary layer,

(*b*) large ω, small ξ, small $\omega\xi$, i.e. within the boundary layer.

Case (a) Outside the boundary layer

In this region we can replace the hyperbolic functions by the appropriate exponential functions, and it follows that when ω is sufficiently large

$$G_0(\xi) \sim 1 + O(e^{-\omega\xi}).$$

Consequently all the required derivates of $G_0(\xi)$ in (4·40) are $O(e^{-\omega\xi})$ and may be ignored. The function $G_0'''(1)$ defined in (4·38) behaves like $2\omega^3$ for large ω and consequently $G_1(\xi)$ and its derivates are $O(e^{-\omega\xi})$. Thus when ω is sufficiently large we can write (4·40) in the form

$$v = \frac{\kappa}{2\omega^2 a}\xi J_1(\eta) + \frac{\kappa J_1(\eta)}{8\omega^6 a}\left\{G_0''''(0) + \frac{\epsilon^* J_0(\eta)}{4}\left[-3G_0''(0) - 5G_1''''(0) - \left(4 - \frac{2}{\omega}\right)G_0'''(0)\right]\ldots\right\}.$$

In general it is easy to show that

$$G_0''(0) = \frac{2\omega^2(S_1 - s_1)(C_1 - c_1)}{S_1 C_1 - s_1 c_1}, \tag{4·41}$$

$$G_0'''(0) = \frac{2\omega^3(2c_1 C_1 + s_1^2 - c_1^2 - S_1^2 - C_1^2)}{S_1 C_1 - s_1 c_1}, \tag{4·42}$$

and when ω is sufficiently large $G_0''(0) \sim 2\omega^2$, $G_0'''(0) \sim -4\omega^3$. Likewise we have $G_1''''(0) \sim 2\omega^2$, and hence

$$v \sim \frac{\kappa}{2\omega^2 a} J_1(\eta)\left\{\xi - \frac{1}{\omega} + \frac{\epsilon^*}{\omega}\left(1 - \frac{3}{2\omega}\right) J_0(\eta) + O(\epsilon^{*2})\right\}. \tag{4·43}$$

When $\epsilon^* = 0$ it will be observed that the formula (4·43) for v reduces to that in (3·38). It will be noted from this formula for v that in addition to having zeros on $r = 0$ and $r = a$ the zonal velocity now vanishes also on the surface

$$\xi = \frac{1}{\omega} - \frac{\epsilon^*}{\omega} J_0(\eta)\left(1 - \frac{3}{2\omega}\right). \tag{4·44}$$

The surface σ defined by (4·44) exists within the fluid where the Bessel function $J_0(\eta)$ is negative, that is, from the first zero of $J_0(\eta)$ at $\eta = 2·4$ to the value $\eta = 3·8$, where $J_1(\eta)$ first vanishes. At $\eta = 3·8$, $J_0(\eta)$ has its minimum value, hence the surface σ has its maximum height above the base at the side of the dishpan and lowers towards the interior. Above σ we have $v > 0$ and below σ we have $v < 0$, that is, 'westerlies' above and 'easterlies' below σ. In order to find out where σ meets the base $\xi = 0$ of the dishpan we must proceed to case (*b*).

Case (b) Inside the boundary layer

Here ξ and $\omega\xi$ are small but ω large, and we can expand the trigonometric and hyperbolic functions which involve $\omega\xi$ in ascending powers of $\omega\xi$. We obtain thereby

$$G_0(\xi) = \omega^2\xi^2 - \tfrac{2}{3}\omega^3\xi^3 + \tfrac{1}{6}\omega^4\xi^4 - \tfrac{1}{90}\omega^6\xi^6 \ldots, \tag{4·45}$$

$$G_1(\xi) = \xi - \omega\xi^2 + \tfrac{1}{3}\omega^2\xi^3 - \tfrac{1}{30}\omega^4\xi^5 + \tfrac{1}{90}\omega^5\xi^6 \ldots, \tag{4·46}$$

the terms to order ξ^3 having been obtained by expanding (4·19) and (4·34) respectively, and the terms of higher order by using the differential equations which G_0 and G_1 satisfy. If we now return to (4·40) and substitute for G_0 and G_1 we obtain

$$v = \frac{\kappa J_1(\eta)}{8\omega^6 a}\left\{(\tfrac{4}{3}\omega^6\xi^3 + \ldots) + \frac{\epsilon^*}{4} J_0(\eta)\,(16\omega^4\xi - \ldots)\right..$$

When $\epsilon^* = 0$ it will be observed that v is of order ξ^3, a result which was obtained in §3 and which is responsible for producing the zero horizontal stress at $\xi = 0$ for that flow. When $\epsilon^* \neq 0$ the height ξ enters as a linear quantity in the ϵ^* term, and hence there will be a finite stress associated with this term. When $\epsilon^* \neq 0$ also the surface $v = 0$ is given approximately by

$$\omega^2\xi^2 = -3\epsilon^* J_0(\eta). \tag{4·47}$$

Hence the surface σ meets the base $\xi = 0$ at the zero $\eta = 2{\cdot}4$ of $J_0(\eta)$ and exists in the region $\eta \geqslant 2{\cdot}4$. Thus the present case confirms case (*a*) that the surface σ exists in $2{\cdot}4 \leqslant \eta \leqslant 3{\cdot}8$.

We now carry out the same investigation for ψ. We have from (4·39)

$$\psi = -\frac{\eta J_1}{4\omega^4 a^2}\left\{G_0 - \tfrac{1}{4}\epsilon^* J_0(\eta)\left[\xi G_0 - 5\xi + G_0\left(4 - \frac{G_0'''(1)}{\omega^4}\right) + 5G_1\right] + O(\epsilon^{*2})\right\}, \tag{4·48}$$

and thus outside the boundary layer we have, for sufficiently large ω,

$$\psi = -\frac{\eta J_1}{4\omega^4 a^2}\left\{1 - \tfrac{1}{4}\epsilon^* J_0(\eta)\left[-4\xi + 4 - \frac{2}{\omega}\right] + O(\epsilon^{*2})\right\}. \tag{4·49}$$

When $\epsilon^* = 0$, ψ is a function of η only and the flow is therefore entirely in surfaces $r =$ constant. When $\epsilon^* \neq 0$ the extra term which is due to $w(\partial v/\partial z)$ produces a radial flow of magnitude

$$u = \frac{\kappa\epsilon^*}{4a\omega^4} J_1(\eta)\, J_0(\eta). \tag{4·50}$$

This u function vanishes at the central axis $r = 0$, is positive in the range $0 < \eta < 2{\cdot}4$, zero again at $\eta = 2{\cdot}4$ and is negative in the range $2{\cdot}4 < \eta < 3{\cdot}8$; in other words, we have an outflow in $0 < \eta < 2{\cdot}4$ and an inflow in $2{\cdot}4 < \eta < 3{\cdot}8$.

The vertical velocity field is obtained from the formula

$$w = \frac{\kappa a^2}{\eta}\frac{\partial\psi}{\partial\eta}, \tag{4·51}$$

and w is accordingly given by

$$w = \frac{\kappa}{4\omega^4}\left\{-J_0 + \epsilon^*\left(-4\xi + 4 - \frac{2}{\omega}\right)(J_0^2 - J_1^2) + O(\epsilon^{*2})\right\}. \tag{4·51 a}$$

When $\epsilon^* = 0$ we reproduce the result contained in (3·25) that the vertical velocity is downwards in $0 \leqslant \eta < 2{\cdot}4$, zero at $\eta = 2{\cdot}4$ and upward in $2{\cdot}4 < \eta < 3{\cdot}8$. The additional term in w when $\epsilon^* \neq 0$ has a linear decrease with height to vanishingly small values near $\xi = 1$, whilst in the radial direction this term produces an upward motion in $0 < \eta < 1{\cdot}44$, a downward motion in $1{\cdot}44 < \eta < 3{\cdot}1$ and an upward motion in $3{\cdot}1 < \eta < 3{\cdot}8$. Thus the upward flow in the outer portion of the dishpan will be augmented by the additional term, and the downward flow near the central axis will be diminished but the region where previously w was zero, namely, at $\eta = 2{\cdot}4$, is now a region of down-flowing currents. It may be significant that this last-mentioned region corresponds to the subtropical regions of descending air in atmospheric motion on the spherical earth. It is of interest also to investigate conditions at the free surface $\xi = 1$. From (4·48) it follows that at the free surface we have

$$u = \frac{\kappa J_1}{4\omega^4 a}\left\{G_0'(1) - \tfrac{1}{4}\epsilon^* J_0(\eta)\left[G_0'(1) - 5 + \left(4 - \frac{G_0'''(1)}{\omega^4}\right)G_0'(1) + 5G_1'(1)\right] + O(\epsilon^{*2})\right\}.$$

It is easily shown that

$$G_0'(1) = -\frac{\omega(S_1-s_1)^2}{S_1C_1-s_1c_1},$$

$$G_0'''(1) = \frac{2\omega^3(C_1-c_1)^2}{S_1C_1-s_1c_1},$$

$$G_1'(1) = \frac{\omega}{S_1C_1-s_1c_1}\left\{C_1^2-c_1^2-\frac{G_0'(1)}{5\omega^2}(S_1^2-s_1^2)+\frac{1}{\omega}(S_1c_1-C_1s_1)\right\},$$

and thus when ω is sufficiently large $G_0'(1) \sim -\omega$, $G_0'''(1) \sim 2\omega^3$, $G_1'(1) \sim \omega\left(1+\frac{1}{5\omega}\right)$. Hence we have

$$u\,|_{\xi=1} = \frac{\kappa J_1}{4\omega^4 a}\{-\omega+\tfrac{1}{2}\epsilon^* J_0(\eta)+O(\epsilon^{*2})\},$$

and thus, as in (3·40), there is a weak inflow at the free surface with $\epsilon^*=0$ which becomes augmented in $2{\cdot}4<\eta<3{\cdot}8$ but diminished in $0<\eta<2{\cdot}4$ when $\epsilon^* \neq 0$.

In the boundary layer near $\xi=0$ we proceed as follows. From (4·48), (4·45) and (4·46) we have

$$\psi = -\frac{\eta J_1 \xi^2}{4\omega^2 a^2}\{1-\epsilon^* J_0(\eta)+O(\epsilon^{*2})\},$$

hence within the boundary layer

$$u = \frac{\kappa \xi J_1(\eta)}{2\omega^2 a}\{1-\epsilon^* J_0(\eta)+O(\epsilon^{*2})\}.$$

Thus u is directed radially outwards when $\epsilon^*=0$, but when $\epsilon^* \neq 0$ the outward flow is diminished in $0<\eta<2{\cdot}4$ and augmented in $2{\cdot}4<\eta<3{\cdot}8$.

We observe finally that the moment of the horizontal stresses at the base about the central axis is given by

$$\int_0^a 2\pi\mu r^2 \frac{\partial v}{\partial z}\bigg|_{z=0} \mathrm{d}r. \tag{4·52}$$

To the order ϵ^* this is evidently proportional to

$$\int_0^{3\cdot 8} \eta^2 J_0(\eta)\, J_1(\eta)\, \mathrm{d}\eta, \tag{4·53}$$

and since this equals $\left[\tfrac{1}{2}\eta^2 J_1^2(\eta)\right]_0^{3\cdot 8}$ it evidently is zero.

5. The effect of the inertia term $\dfrac{u}{r}\dfrac{\partial}{\partial r}(vr)$

The effects of this term are investigated in exactly the same way as the term $w(\partial v/\partial z)$ of § 4. The v factor of this term is replaced by v^*, and the equations governing the flow are taken to be (4·1), (4·3), (4·4), (4·5) together with

$$\rho_0 u\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv^*)+2\Omega\right\} = \mu\frac{\partial^2 v}{\partial z^2}. \tag{5·1}$$

Using the definition of v^* in (3·42) this becomes

$$u\left\{2\omega^2+\frac{\epsilon}{2\omega^2}\left(\xi-\frac{1}{\omega}\right)J_0(\eta)\right\} = \frac{\partial^2 v}{\partial \xi^2}. \tag{5·2}$$

Equation (4·8) can be arranged in the form

$$-2\omega^2\frac{\partial v}{\partial \xi} = -\frac{\kappa}{\sinh a}\cosh a(1-\xi)\, J_1(\beta r)+\frac{\partial^3 u}{\partial \xi^3}, \tag{5·3}$$

and when v is eliminated between the equations (5·2) and (5·3) we obtain

$$\frac{\partial^4 u}{\partial \xi^4}+4\omega^4 u+\epsilon\left(\xi-\frac{1}{\omega}\right)J_0(\eta)\, u = -\frac{\kappa a}{\sinh a}\sinh a(1-\xi)\, J_1(\eta). \tag{5·4}$$

It is convenient to introduce the Stokes stream function ψ defined in (4·9) since we can then make use of the results of § 4. The equation for ψ is

$$\frac{\partial^5 \psi}{\partial \xi^5}+4\omega^4\frac{\partial \psi}{\partial \xi}+\epsilon\left(\xi-\frac{1}{\omega}\right)J_0(\eta)\frac{\partial \psi}{\partial \xi} = \frac{\sinh a(1-\xi)}{\sinh a}\eta J_1(\eta). \tag{5·5}$$

Substituting the ϵ^* expansion of (4·12) we obtain

$$\frac{\partial^5 \psi_0}{\partial \xi^5}+4\omega^4\frac{\partial \psi_0}{\partial \xi} = \frac{\sinh a(1-\xi)}{\sinh a}\eta J_1(\eta), \tag{5·6}$$

$$\frac{\partial^5 \psi_1}{\partial \xi^5}+4\omega^4\frac{\partial \psi_1}{\partial \xi} = -4\omega^4\left(\xi-\frac{1}{\omega}\right)J_0(\eta)\frac{\partial \psi_0}{\partial \xi}, \tag{5·7}$$

and so on. With the same conditions (4·15) and (4·30) imposed upon ψ_0 and upon the zonal velocity it follows that the function ψ_0 is the same as in (4·33), namely,

$$\psi_0 = -\frac{1}{4\omega^4 a^2}\eta J_1(\eta)\, G_0(\xi), \tag{5·8}$$

so that (5·7) becomes

$$\frac{\partial^5 \psi_1}{\partial \xi^5}+4\omega^4\frac{\partial \psi_1}{\partial \xi} = \frac{1}{a^2}\eta J_0(\eta)\, J_1(\eta)\left(\xi-\frac{1}{\omega}\right)\frac{dG_0}{d\xi}. \tag{5·9}$$

It may be verified that the particular solution for ψ_1 arising from (5·9) is

$$\psi_1' = \frac{1}{32\omega^4 a^2}\eta J_1(\eta)\, J_0(\eta)\left\{-\xi^2 G_0'+5\xi G_0-5\xi+\frac{2}{\omega}\xi G_0'\right\}, \tag{5·10}$$

and that the complete solution for ψ_1 is given by

$$\psi_1 = \psi_1'+F_1(\eta)\, G_0(\xi)+\alpha_1\sinh\omega\xi\sin\omega\xi+\beta_1\sinh\omega\xi\sin\omega(1-\xi)+\gamma_1\sinh\omega(1-\xi)\sin\omega\xi, \tag{5·11}$$

where

$$\left.\begin{aligned}\alpha_1 S_1 s_1 &= \frac{1}{32\omega^4 a^2}\eta J_1 J_0\left\{5+G_0'(1)\left(1-\frac{2}{\omega}\right)\right\},\\ \beta_1 s_1+\gamma_1 S_1 &= \frac{1}{32\omega^4 a^2}\eta J_1 J_0\left\{\frac{5}{\omega}\right\},\\ \alpha_1 C_1 c_1-\beta_1 C_1-\gamma_1 c_1 &= \frac{1}{32\omega^4 a^2}\eta J_1 J_0\left\{\frac{\left(1-\frac{2}{\omega}\right)G_0'''(1)-8G_0'(1)}{2\omega^2}\right\}.\end{aligned}\right\} \tag{5·12}$$

Consider now equation (5·4). When it is integrated with respect to ξ we obtain

$$-2\omega^2 v = \frac{\kappa}{a\sinh a}\sinh a(1-\xi)\, J_1(\beta r)+\frac{\partial^2 u}{\partial \xi^2}-2\omega^2\chi(r),$$

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where $\chi(r)$ is an arbitrary function of r. Replacing u by its expression in terms of ψ, (4·9), we obtain

$$v = -\frac{\kappa}{2\omega^2 a \sinh a} \sinh a(1-\xi)\, J_1(\eta) + \frac{\kappa a}{2\omega^2 \eta} \frac{\partial^3 \psi}{\partial \xi^3} + \chi(r). \tag{5·13}$$

The function $\chi(r)$ is chosen so that $v = 0$ at $\xi = 0$, hence

$$\chi(r) = \frac{\kappa}{2\omega^2 a} J_1(\eta) - \frac{\kappa a}{2\omega^2 \eta} \left(\frac{\partial^3 \psi}{\partial \xi^3}\right)_{\xi=0}. \tag{5·14}$$

The upper boundary condition $\partial v/\partial \xi = 0$ at $\xi = 1$ leads to the same results as in (4·30) and (4·31). The former is satisfied by the ψ_0 of (5·8), provided $\sinh a$ is replaced by a. The latter leads to the definition of $F_1(\eta)$. Using (5·11) and (4·31) we have

$$\frac{1}{32\omega^4 a^2} \eta J_1 J_0 \left\{-\xi^2 G_0' + 5\xi G_0 - 5\xi + \frac{2}{\omega} \xi G_0'\right\}^{\text{iv}}_{\xi=1} + F_1(\eta)\, G_0^{\text{iv}}(1) - 4\omega^4 \alpha_1 S_1 s_1 = 0. \tag{5·15}$$

If we substitute for α_1 from (5·12) it follows that

$$F_1(\eta) = \frac{1}{4\omega^4 a^2} \eta J_1 J_0 \left\{1 - \frac{G_0'''(1)}{4\omega^4} - \frac{1}{\omega}\right\}. \tag{5·16}$$

Consequently the complete solution for ψ_1 is given by

$$\psi_1 = \frac{1}{32\omega^4 a^2} \eta J_1 J_0 \left\{-\xi^2 G_0' + 5\xi G_0 - 5\xi + \frac{2}{\omega} \xi G_0' + 8\left[1 - \frac{G_0'''(1)}{4\omega^4} - \frac{1}{\omega}\right] G_0(\xi) + H_1(\xi)\right\}, \tag{5·17}$$

where $$H_1(\xi) = \alpha_1' \sinh \omega\xi \sin \omega\xi + \beta_1' \sinh \omega\xi \sin \omega(1-\xi) + \gamma_1' \sinh \omega(1-\xi) \sin \omega\xi, \tag{5·18}$$

and α_1', β_1' and γ_1' are the solutions of the equations

$$\left.\begin{aligned} \alpha_1' S_1 s_1 &= 5 + G_0'(1)\left(1 - \frac{2}{\omega}\right), \\ \beta_1' s_1 + \gamma_1' S_1 &= \frac{5}{\omega}, \\ \alpha_1' C_1 c_1 - \beta_1' C_1 - \gamma_1' c_1 &= \frac{1}{2\omega^2}\left\{\left(1 - \frac{2}{\omega}\right) G_0'''(1) - 8G_0'(1)\right\}. \end{aligned}\right\} \tag{5·19}$$

When ω is sufficiently great $G_0'(1) \sim -\omega$, $G_0'''(1) \sim 2\omega^3$ and thus (5·17) becomes approximately

$$\psi_1 \sim \frac{1}{4\omega^4 a^2} \eta J_0(\eta)\, J_1(\eta) \left(1 - \frac{3}{2\omega}\right), \tag{5·20}$$

provided the position is outside the boundary layer. Since this asymptotic value for ψ_1 is independent of ξ it follows that the radial velocity arising from it is zero. The vertical velocity distribution using (4·51 a) will be (outside the boundary layer)

$$w = \frac{\kappa \epsilon^*}{4\omega^4} \left(1 - \frac{3}{2\omega}\right) (J_0^2 - J_1^2), \tag{5·21}$$

and thus, as in § 4, we have ascending currents in $0 \leqslant \eta < 1·4$ and $2·4 < \eta < 3·8$, with descending currents in $1·4 < \eta < 2·4$. Thus as far as the field of vertical velocity is concerned the term $\frac{u}{r} \frac{\partial}{\partial r}(vr)$ tends to augment the term $w(\partial v/\partial z)$, the magnitudes of the vertical velocity being largely the same.

We consider now the zonal velocity. From (5·13) and (5·14) we have

$$v = \frac{\kappa}{2\omega^2 a} J_1(\eta)\left\{\xi - \frac{1}{4\omega^4}\left[G_0'''(\xi) - G_0'''(0)\right]\right\} + \frac{2\kappa a \epsilon^* \omega^2}{\eta}\left\{\frac{\partial^3\psi_1}{\partial\xi^3} - \left(\frac{\partial^3\psi_1}{\partial\xi^3}\right)_0\right\}, \tag{5·22}$$

and using (5·17) the second bracket becomes

$$\frac{\kappa\epsilon^*}{64\omega^6 a} J_1 J_0\Big[-\xi^2 G_0^{\text{iv}}(\xi) - 6\xi G_0'''(\xi) + 9G_0''(\xi) - 9G_0''(0) + 5\xi G_0'''(\xi) + \frac{2}{\omega}\xi G_0^{\text{iv}}(\xi) + \frac{6}{\omega}\{G_0''(\xi) - G_0''(0)\}$$
$$+ H_1'''(\xi) - H_1'''(0) + 8\left(1 - \frac{G_0'''(1)}{4\omega^4} - \frac{1}{\omega}\right)(G_0'''(\xi) - G_0'''(0))\Big].$$

If the point ξ is outside the boundary layer, then as ω becomes large this term behaves like

$$\frac{\kappa\epsilon^*}{64\omega^6 a} J_1 J_0\Big[-9G_0''(0) - \frac{6}{\omega}G_0''(0) - H_1'''(0) - 8\left(1 - \frac{G_0'''(1)}{4\omega^4} - \frac{1}{\omega}\right)G_0'''(0)\Big].$$

If we now use the results (4·41), (4·42) and (4·38) this becomes

$$\frac{\kappa\epsilon^*}{64\omega^6 a} J_1 J_0\Big[-18\omega^2 - 12\omega - 8\left(1 - \frac{3}{2\omega}\right)(-4\omega^3)\Big].$$

The leading term will be $\kappa\epsilon^* J_1 J_0/2\omega^3 a$, and consequently the expression for v becomes

$$v = \frac{\kappa}{2\omega^2 a} J_1(\eta)\left\{\xi - \frac{1}{\omega} + \frac{\epsilon^*}{\omega} J_0(\eta)\right\}. \tag{5·23}$$

If we compare (5·23) and (4·43) it will be observed that the effect of the term $\frac{u}{r}\frac{\partial}{\partial r}(vr)$ is precisely the same as the $w(\partial v/\partial z)$ term in the formation of easterly and westerly zonal flow patterns. Hence when the two terms are combined the resulting $v = 0$ surface will be

$$\xi = \frac{1}{\omega} - \frac{2\epsilon^*}{\omega} J_0(\eta) \quad (2·4 \leqslant \eta \leqslant 3·8) \tag{5·24}$$

outside the boundary layer. The moment of the surface stresses about $r = 0$ is zero as in § 4.

6. The effect of the term v^2/r

We now introduce the centrifugal term v^2/r into the radial equation of motion and consider the effect of this term upon the flow of § 3. The equations are simplified by taking vv^*/r in place of v^2/r. We then have

$$-2\Omega\rho_0 v - \frac{\rho_0 v v^*}{r} = -\frac{\partial p}{\partial r} + \mu\frac{\partial^2 u}{\partial z^2}, \tag{6·1}$$

$$2\Omega\rho_0 u = \mu\frac{\partial^2 v}{\partial z^2}, \tag{6·2}$$

and the remaining equations are identical with those in (4·3) to (4·5). When we use the ψ function defined in (4·9) we obtain from (6·2)

$$\frac{\partial^2 v}{\partial\xi^2} = -\frac{2\kappa\omega^2 a}{\eta}\frac{\partial\psi}{\partial\xi}$$

by introducing the usual non-dimensional quantities. Upon integration and use of the boundary condition $\psi = 0$, $\partial v/\partial\xi = 0$ at $\xi = 1$ we obtain

$$\frac{\partial v}{\partial \xi} = -\frac{2\kappa\omega^2 a}{\eta}\psi. \tag{6·3}$$

Equation (6·1) may be written in the form

$$2\omega^2 v + \frac{\kappa\rho_0 h}{2\omega^2\mu} v\left(\xi - \frac{1}{\omega}\right)\frac{1}{\eta}J_1(\eta) = \frac{g\alpha h^3}{\mu}\frac{\partial T}{\partial r} + \frac{\kappa a}{\eta}\frac{\partial^3\psi}{\partial\xi^3},$$

and by eliminating v between this equation and (6·3) we obtain

$$\frac{\partial^4\psi}{\partial\xi^4} + 4\omega^4\psi + \epsilon\left(\xi - \frac{1}{\omega}\right)\frac{1}{\eta}J_1(\eta)\,\psi = -\frac{\cosh a(1-\xi)}{a\sinh a}\,\eta J_1(\eta). \tag{6·4}$$

We investigate a solution of (6·4) of the form (4·12) and consequently the successive ψ_s functions in that development satisfy

$$\frac{\partial^4\psi_0}{\partial\xi^4} + 4\omega^4\psi_0 = -\frac{1}{a\sinh a}\cosh a(1-\xi)\,\eta J_1(\eta), \tag{6·5}$$

$$\frac{\partial^4\psi_1}{\partial\xi^4} + 4\omega^4\psi_1 = -4\omega^4\left(\xi - \frac{1}{\omega}\right)\frac{1}{\eta}J_1(\eta)\,\psi_0, \tag{6·6}$$

and so on. The function ψ_0 satisfies the same conditions as in §4 and hence is given by (4·33), so that (6·6) becomes

$$\frac{\partial^4\psi_1}{\partial\xi^4} + 4\omega^4\psi_1 = \frac{1}{a^2}\left(\xi - \frac{1}{\omega}\right)J_1^2(\eta)\,G_0(\xi). \tag{6·7}$$

It is easily shown that the complete solution of (6·7) is given by

$$\psi_1 = \frac{1}{32\omega^4 a^2}J_1^2(\eta)\left\{-\xi^2 G_0' + 3\xi G_0 + \frac{2}{\omega}\xi G_0' + 5\xi - \frac{8}{\omega}\right\} + \Psi_1, \tag{6·8}$$

where

$$\Psi_1 = A\sinh\omega\xi\,\sin\omega\xi + B\sinh\omega\xi\,\sin\omega(1-\xi) + C\sinh\omega(1-\xi)\,\sin\omega\xi + D\sinh\omega(1-\xi)\,\sin\omega(1-\xi). \tag{6·9}$$

In order to satisfy the conditions (4·15), which apply also to ψ_1, we choose A, B, C and D so that

$$DS_1 s_1 = \frac{1}{32\omega^4 a^2}J_1^2\left\{\frac{8}{\omega}\right\},$$

$$AS_1 s_1 = \frac{1}{32\omega^4 a^2}J_1^2\left\{G_0'(1)\left(1 - \frac{2}{\omega}\right) + \frac{8}{\omega} - 5\right\},$$

$$Bs_1 + CS_1 - D(C_1 s_1 + S_1 c_1) = \frac{1}{32\omega^4 a^2}J_1^2\left\{-\frac{5}{\omega}\right\},$$

$$AC_1 c_1 - BC_1 - Cc_1 + D = \frac{1}{32\omega^4 a^2}J_1^2\left\{\frac{G_0'''(1)\left(1 - \frac{2}{\omega}\right) - 4G_0'(1)}{2\omega^2}\right\}.$$

From (6·3) we have the following formula for the zonal velocity:

$$v = -\frac{2\kappa\omega^2 a}{\eta}\int_0^\xi \psi\,\mathrm{d}\xi$$

$$= -\frac{2\kappa\omega^2 a}{\eta}\int_0^\xi (\psi_0+\epsilon^*\psi_1+\dots)\,\mathrm{d}\xi. \tag{6·10}$$

It is sufficient to investigate the above solutions for large ω outside the boundary layer. In this case we have

$$\psi_0 \sim -\frac{1}{4\omega^4 a^2}\eta J_1(\eta), \tag{6·11}$$

$$\psi_1 \sim \frac{1}{32\omega^4 a^2}\left(8\xi-\frac{8}{\omega}\right)J_1^2(\eta). \tag{6·12}$$

Since $u = -\dfrac{\kappa a}{\eta}\dfrac{\partial\psi}{\partial\xi}$ it follows that the additional radial flow due to the v^2/r term will be $-\kappa J_1^2(\eta)/4\omega^4\eta$, which is, as expected, everywhere inwards. Since $w = \dfrac{\kappa a^2}{\eta}\dfrac{\partial\psi}{\partial\eta}$ it follows that the additional vertical flow due to the v^2/r term will be $\kappa\left(\xi-\dfrac{1}{\omega}\right)J_1J_1'/2\omega^4\eta$, which is positive in the range $0<\eta<1·85$ and negative in the range $1·85<\eta<3·8$.

When (6·12) and (6·11) are substituted in (6·10) it becomes approximately

$$v = -\frac{2\kappa\omega^2 a}{\eta}\left\{-\frac{1}{4\omega^4 a^2}\left(\xi-\frac{1}{\omega}\right)\eta J_1(\eta)+\frac{\epsilon^*}{4\omega^4 a^2}\left(\tfrac{1}{2}\xi^2-\frac{\xi}{\omega}\right)J_1^2(\eta)+O(\epsilon^{*2})\right\}.$$

Outside the boundary layer ($\xi \gg 1/\omega$) therefore the additional term produces an easterly flow, but this can nowhere exceed the main westerly flow and serves only to reduce the westerly flow. It is evident from (6·10) that this centrifugal term does not contribute to the surface stresses.

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